A CYCLOTOMIC FAMILY OF THIN HYPERGEOMETRIC MONODROMY GROUPS IN $\mathrm{Sp}_4(\mathbb{R})$

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Abstract. We exhibit an infinite family of discrete subgroups of $\mathbf{Sp}_4(\mathbb{R})$ which have a number of remarkable properties. Our results are established by showing that each group plays ping-pong on an appropriate set of cones. The groups arise as the monodromy of hypergeometric differential equations with parameters $(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N})$ at infinity and maximal unipotent monodromy at zero, for any integer $N \geq 3$.

Contents

1. Introduction

The monodromy of hypergeometric differential equations has been actively studied for a long time. A historical overview going back to the 19th century can be found in the book of Gray [Gra08], and the more recent developments relevant to the present text started with the work of Beukers–Heckman [BH89] who analyzed the basic features of the monodromy groups of hypergeometric equations on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. In particular, they described the Zariski closure of the discrete groups which arise.

A more refined question about the monodromy group is what is its relation to the ambient arithmetic lattice. The most interesting case is when the monodromy representation identifies the fundamental group of the base (in an orbifold sense) with the corresponding arithmetic lattice, and this leads to uniformization of algebraic manifolds by domains. A representative example is the congruence subgroup $\Gamma(2)$ of $\mathbf{SL}_2(\mathbb{Z})$. In a different direction, the representation can surject (often with large kernel) onto a finite index subgroup of the arithmetic group. Results of this nature have been obtained recently by Singh–Venkataramana [SV14]. Finally, the image of the representation can be an infinite index subgroup of the lattice, and this is called the "thin" case[1]. This is the case of interest to us.

An infinite family of thin monodromy groups has been obtained by Fuchs, Meiri, and Sarnak [FMS14]. These are discrete subgroups of the indefinite orthogonal group $\mathbf{SO}_{1,n}(\mathbb{R})$, finitely many for each n, and arbitrarily large n. By different methods, Brav and Thomas [BT14] exhibited 7 parameters for which the monodromy group is thin in $\mathbf{Sp}_4(\mathbb{Z}) \subset \mathbf{Sp}_4(\mathbb{R})$.

In this text, we extend the methods of Brav–Thomas and exhibit an infinite family of parameters for which the monodromy yields discrete subgroups in $\mathbf{Sp}_4(\mathbb{R})$. These are moreover thin, when viewed inside appropriately chosen lattices. Note that as our ambient dimension is fixed at 4, the matrix entries of the groups will necessarily lie in number fields of increasing size.

[1] So far, all known thin examples appear also to be (essentially) injective.

Parameters for hypergeometric equations. We will consider rank 4 hypergeometric groups with maximal unipotent monodromy at zero (see Section 2 for more background on hypergeometric equations). This leads to the differential operator

$$D^4 - z(D+\alpha_1)(D+\alpha_2)(D+\alpha_3)(D+\alpha_4) \quad D = z\partial_z$$

which has regular singular points at $0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$. The α-parameters that we consider are $\left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right)$ for $N \geq 4$. We let $\Gamma_N \subset \mathbf{Sp}_4(\mathbb{R})$ be the monodromy group of the equation, and $P\Gamma_N \subset \mathbf{PSp}_4(\mathbb{R})$ its image in the projectivized symplectic group. Let R be the monodromy at infinity and T the monodromy around 1.

1.0.1. Theorem (Thin, discrete monodromy)**.** *The projective monodromy group is isomorphic, as an abstract group, to:*

$$P\Gamma_N \cong \left\langle R, T \,|\, R^N = 1 \right\rangle$$

Furthermore, it is a discrete subgroup of $\mathbf{PSp}_4(\mathbb{R})$.
The full monodromy group Γ_N is isomorphic to $P\Gamma_N$ if N is odd, and is a $\mathbb{Z}/2$ central extension if N is even. It is also a discrete subgroup of $\mathbf{Sp}_4(\mathbb{R})$.
Furthermore, denote by Y_N the orbifold $\mathbb{P}^1 \setminus \{0,1\}$ with an orbifold point of order N at infinity. Then the monodromy representation is an isomorphism of $\pi_1^{orb}(Y_N)$ and $P\Gamma_N$.

Proof. In §2.2 we construct a group generated by three reflections $\tilde{\Gamma}_N$ that contains Γ_N with index 2. Let the reflections be A, B, C and Γ_N is mapped to $\tilde{\Gamma}_N$ via $R \mapsto BC$ and $T \mapsto AB$. In Theorem 3.2.5, we show that $\tilde{\Gamma}_N$ acts on a set of $2N$ cones in $\mathbb{P}(\mathbb{R}^4)$ in a (generalized) ping-pong manner. It follows that for its image in the projective group we have:

$$P\tilde{\Gamma}_N \cong \left\langle A, B, C \,|\, A^2 = B^2 = C^2 = 1, (BC)^N = 1 \right\rangle.$$

Then $\tilde{\Gamma}_N$ is either a $\mathbb{Z}/2$-extension of $P\tilde{\Gamma}_N$ or isomorphic to it, according to whether $R^N = -\mathbf{1}$ (N is even) or $R^N = \mathbf{1}$ (N is odd). □

On thinness. According to the customary definition, see e.g. the survey of Sarnak [Sar14], a thin group is one which is of infinite index in an arithmetic lattice. Let us explain how this arises in our context.

Let $\mathbb{Q}(\zeta_{2N})$ be an extension of $\mathbb{Q}$ obtained by adjoining a $2N$-th root of unity, $\mathbb{Q}(\zeta_{2N})^+$ its totally real subfield, and $\mathcal{O}_{2N}^+$ the corresponding ring of integers. Plainly from the definitions of hypergeometric groups (see e.g. [BH89]) it follows that the monodromy matrices can be expresses with entries in $\mathcal{O}_{2N}^+$. However, the multiplicatively invertible residue classes $(\mathbb{Z}/2N)^\times$ have a Galois action on the roots of unity, and we can identify those with $\left(\frac{1}{2N}\mathbb{Z}\right)/\mathbb{Z}$, or rationals in $[0,1)$ with denominator $2N$. Then a subgroup will stabilize our given four-tuple $\alpha_\bullet^{(0)} = \left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right)$. We also have the orbit of our four-tuple under this multiplicative action, with representatives (all mod 1) $\alpha_\bullet^{(i)}$, say a total of k distinct representatives.

This defines a subfield $K_N \subset \mathbb{Q}(\zeta_{2N})^+$ such that the Galois group of $\mathbb{Q}(\zeta_{2N})/K_N$ is precisely the stabilizer in $(\mathbb{Z}/2N)^\times$ of our original four-tuple (note that -1 stabilizes our four-tuple so the subfield is totally real). If we denote by $\mathcal{O}_{K_N}$ the ring of integers in K_N then our monodromy group Γ_N embeds in $\mathbf{Sp}_4(\mathcal{O}_{K_N})$. This last group is an arithmetic lattice in $\mathbf{Sp}_4(\mathbb{R})^k$ (a product of k copies of $\mathbf{Sp}_4(\mathbb{R})$) where k is the number of four-tuples obtained by the multiplicative action on our original one. The projection of $\Gamma_N \hookrightarrow \mathbf{Sp}_4(\mathcal{O}_{K_N}) \hookrightarrow \mathbf{Sp}_4(\mathbb{R})^k$ to any of the $\mathbf{Sp}_4(\mathbb{R})$-factors yields the Galois-conjugate local systems of our original one.

The monodromy group Γ_N is visibly discrete in the product $\mathbf{Sp}_4(\mathbb{R})^k$ since it is contained in the discrete lattice there. But Theorem 1.0.1 implies that Γ_N is in fact discrete when projected to one of the factors. This is similar to the classical constructions of Deligne–Mostow [DM86] of non-arithmetic lattices in $\mathbf{SU}(1,n)$, with the difference that our group $\mathbf{Sp}_4(\mathbb{R})$ is of higher rank.

Let us remark that Veech groups, which arise in Teichmüller dynamics, also yield lattices in $\mathbf{SL}_2(\mathbb{R})$ but also yield thin groups in Hilbert modular groups, which are arithmetic lattices inside products of $\mathbf{SL}_2(\mathbb{R})$. The relation between these lattices and Hodge theory was investigated by Möller [MÖ6], and in higher rank in Teichmüller dynamics in [Fil16]. See also. McMullen's recent investigation [McM20] in this direction, and Zorich's survey [Zor06] for further background in Teichmüller dynamics.

Let us finally remark that the $\mathbb{R}$-Zariski density of Γ_N inside $\mathbf{Sp}_4(\mathbb{R})^k$ follows from the combination of the results of Beukers–Heckman [BH89], which establish Zariski density in each factor separately, and Goursat's lemma in group theory (combined with the fact that $\mathbf{Sp}_4$ is simple).

Numerical experiments. Our work started from observations on numerical represions of the action of monodromy groups on the cones that can be accessed at `https://gitlab.com/fougeroc/ping-pong`. We have also used symbolic computation tools from SageMath [Sag20] and our final worksheet can be found at `ADDRESS` which can be useful for the reader to follow our computations.

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2. Background on hypergeometric groups

Outline of section. In §2.1 we recall some basic definitions regarding hypergeometric differential equations and their monodromy. Next, in §2.2 we extend with index 2 the monodromy group to make it generated by reflections.

2.1. Notation

We recall here some standard facts on hypergeometric groups. See [BH89] or [Yos97] for further background.

2.1.1. Setup. Fix two n-tuples of reals $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ subject to the normalizations $\alpha_i \in [0, 1)$ and $\beta_i \in (0, 1]$. Note that most classical normalizations, which involve explicit hypergeometric functions, take $\beta_n = 1$, and in our case we will take $\beta_i = 1, \forall i$ to ensure maximal unipotent monodromy at 0 (another popular normalization and different expressions for the differential operators are related to ours by $\beta_i \mapsto 1-\beta_i$). Let also $a_i := \exp(2\pi\sqrt{-1}\alpha_i)$ and $b_i = \exp(2\pi\sqrt{-1}\beta_i)$ be (unit) complex numbers.

2.1.2. Differential operator and monodromy. Consider the differential operator

$$D_{\alpha,\beta} := \prod_{i=1}^{n}(D + \beta_i - 1) - z\prod_{i=1}^{n}(D + \alpha_i) \quad D := z\partial_z$$

In $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ its solutions form a local system $\mathbb{V}(\alpha, \beta)$ of rank n, which we will call the hypergeometric local system. Let g_0, g_1, g_∞ be the monodromy matrices of this local system, along paths as described in Figure 2.1.3. Then their conjugacy classes are determined by the following conditions on the characteristic polynomials and ranks:

$$\det(t - g_\infty) = \prod_i (t - a_i)$$
$$\det(t - g_0^{-1}) = \prod_i (t - b_i)$$
$$\mathrm{rk}(g_1 - \mathbf{1}) = 1 \quad \det(g_1) = \exp\left(2\pi\sqrt{-1}\sum(\beta_i - \alpha_i)\right)$$

with the convention that whenever there are repeated roots, there is only one Jordan block.

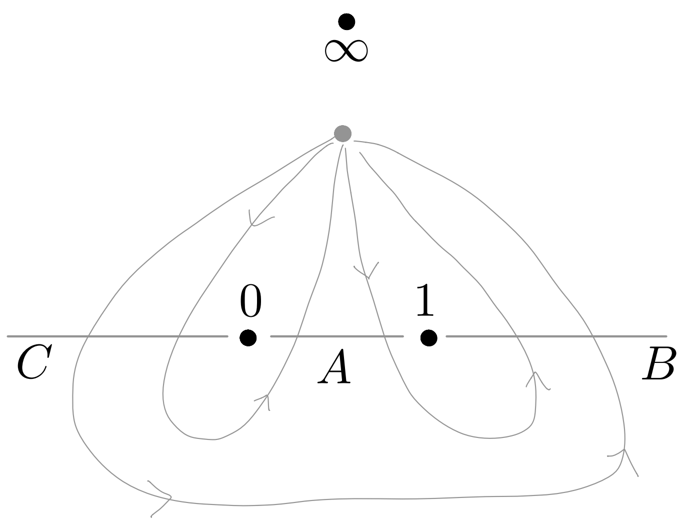

FIGURE 2.1.3. The paths along which we parallel-transport solutions

2.1.4. Rigidity of the local system. Assuming that $\alpha_i - \beta_j \notin \mathbb{Z}$ for any i, j, the local system $\mathbb{V}(\alpha, \beta)$ is irreducible. Furthermore, any local system on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ which has the same conjugacy classes of monodromy matrices around the missing points is isomorphic to the hypergeometric local system. In particular, to verify that an explicit representation of the free group on two letters yields a hypergeometric local system, it suffices to consider the corresponding conjugacy classes of the monodromy matrices around the removed points.

2.1.5. Thin cyclotomic parameters. We will consider the family of hypergeometric groups with parameters

$$\begin{aligned} \beta_\bullet &: \quad (1,1,1,1) \\ \alpha_\bullet &: \quad \left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right) \qquad N \geq 3 \end{aligned}$$

Note that we have the linear equation $-\alpha_1 + 3\alpha_2 = 1$. When working with rotation matrices, we will make use of the parameters:

$$\begin{aligned} \mu_1 &= 2\pi\alpha_1 = \frac{N-3}{N}\pi \\ \mu_2 &= 2\pi\alpha_2 = \frac{N-1}{N}\pi \end{aligned} \tag{2.1.6}$$

2.2. Reflection structure

As stated in §2.1.4, in order to verify that a certain representation is the monodromy of a hypergeometric group, it suffices to consider the conjugacy classes of the corresponding matrices. In this section, we will enlarge (with index 2) our hypergeometric groups to groups generated by reflections. This structure arises because our parameters are real, hence the hypergeometric differential equation has a complex conjugation symmetry and its solutions can be Schwarz-reflected across the real axis.

2.2.1. Abbreviations. To keep formula sizes manageable, we will use the following abbreviations:

$$\begin{aligned} c_1 &:= \cos(\mu_1) \qquad & s_1 &:= \sin(\mu_1) \\ c_2 &:= \cos(\mu_2) \qquad & s_2 &:= \sin(\mu_2) \end{aligned} \tag{2.2.2}$$

where the parameters μ_1, μ_2 are introduced in Eqn. (2.1.6). Their specific numerical values will not be relevant until we reach the calculations with rotated vectors in Section 4.

It will also be convenient to introduce the shorthands:

$$r_1 := \frac{2(c_1-1)^2}{s_1(c_1-c_2)} \qquad r_2 := \frac{2(c_2-1)^2}{s_2(c_2-c_1)} \tag{2.2.3}$$

2.2.4. The reflection matrices. With these preparations, define:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -r_1 & 1 & -r_1 & 0 \\ 0 & 0 & -1 & 0 \\ -r_2 & 0 & -r_2 & 1 \end{bmatrix} B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} C = \begin{bmatrix} -c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & -c_2 & s_2 \\ 0 & 0 & s_2 & c_2 \end{bmatrix}$$

Define also a symplectic pairing on $\mathbb{R}^4$ by the following matrix:

$$J = \begin{bmatrix} 0 & r_2 & 0 & 0 \\ -r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1 \\ 0 & 0 & -r_1 & 0 \end{bmatrix} \tag{2.2.5}$$

2.2.6. Properties of the reflection matrices. For $M \in \{A, B, C\}$, we have that

$$M^t J M = -J$$

or in other words, the above matrices satisfy $\langle Mv, Mw \rangle = -\langle v, w \rangle$ for vectors $v, w \in \mathbb{R}^4$ and $\langle v, w \rangle := v^t J w$ the symplectic pairing. It also follows from the formulas that

$$A^2 = B^2 = C^2 = 1.$$

Let us verify that if we define the monodromy matrices of a local system using the above reflections, as described in Figure 2.1.3, we obtain a hypergeometric group with parameters specified in §2.1.5.

It is immediate that the matrix BC is block-diagonal consisting of rotation matrices by angles μ_1, μ_2, so the conjugacy class at infinity is correct. It is immediate also that the matrix BA of monodromy around 1 is such that $BA - \mathbf{1} = B(A - B)$ is of rank 1. The only necessary calculation is that AC is a maximally unipotent matrix.

One could check it by a tedious and explicit calculation from the above formulas. A shortcut in computations it to use the vectors generating the cone $\mathcal{C}_0$ defined by Eqn. (3.4.6), see also Eqn. (3.2.3) for which vectors are v_i. Then two readily verified properties yield the result. First, one checks that each column vector is an eigenvector of A, with eigenvalue $(-1)^{i+1}$ for v_i. Next, one verifies that C satisfies $Cv_i = (-1)^{i+1} v_i + \sum_{j>i} c_i^j v_j$, i.e. C respects the filtration induced by the vectors v_i. It then follows that CA is a maximally unipotent matrix preserving the filtration induced by the cone vectors.

3. Cones and ping-pong

Outline of section. We describe in §3.1 the hyperbolic triangle reflection groups that give the fundamental group of the orbifold which is the basis for our analysis. Next, in §3.2 we describe the abstract properties of the cones that are used for the ping-pong argument. Based on these abstract properties we explain in §3.3 how to reduce the proof of the ping-pong property to certain explicit calculations. Finally in §3.4 we give the explicit formula for the cone and verify that it has the properties that we used. This reduces the calculations to an explicit analysis in Section 4.

3.1. Triangle reflection groups

3.1.1. Setup. Fix an integer $N \geq 4$. We will be interested in the group given by the generators and relations:

$$G_N := \left\langle a, b, c \,|\, a^2 = b^2 = c^2 = 1, (bc)^N = 1 \right\rangle$$

It is transparent that it acts on the hyperbolic plane such that a, b, c are reflections in geodesics, with the geodesics for b and c forming an angle of $\frac{\pi}{2N}$ and the geodesic for a going between the (nearest) endpoints of the geodesics for b and c. An illustration in the disc model is provided in Figure 3.1.2.

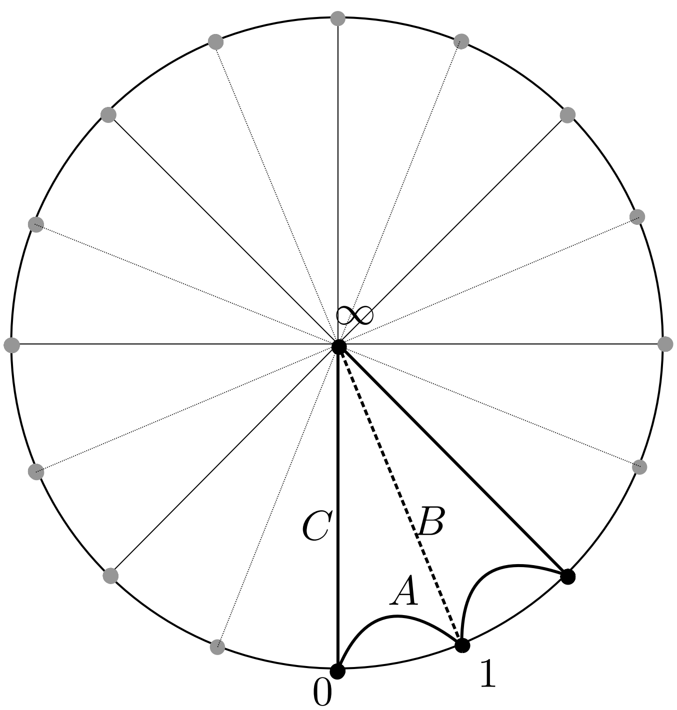

Figure 3.1.2. Fundamental domain for the triangle group action.

3.1.3. Linear and projective representation. Recall that our basic angles from Eqn. (2.1.6) are:

$$(\mu_1, \mu_2) = \left(2\pi\frac{N-3}{2N}, 2\pi\frac{N-1}{2N}\right).$$

and that we defined the matrices A, B, C in §2.2.4. Define the group generated by them:

$$\tilde{\Gamma}_N := \langle A, B, C \rangle \subset \mathbf{GSp}_4(\mathbb{R})$$

They are in the general symplectic group, i.e. $\langle gv, gw \rangle = \chi(g) \langle v, w \rangle$ for a character $\chi \colon \mathbf{GSp}_4 \to \mathbb{G}_m$ and where $\langle -, - \rangle$ is the symplectic pairing.

Let $R := BC$ be the block rotation matrix by the corresponding angles μ_i. We then have the following basic dichotomy:

- N **is odd:** Then the order of R is N and $-\mathbf{1}$ is not in Γ_N.
- N **is even:** Then the order of R is $2N$ and $-\mathbf{1}$ is in Γ_N. Specifically $R^N = -\mathbf{1}$.

Indeed, we have that $gcd(N-1, 2N) = gcd(N-1, 2)$ which is 2 or 1, according to whether N is odd or even.

In both cases we will consider only the projective action of Γ_N, so let:

$$P\tilde{\Gamma}_N := \text{image of } \tilde{\Gamma}_N \subset \mathbf{PGSp}_4(\mathbb{R}).$$

It follows that independently of the parity of N we have the representation

$$\begin{gathered} G_N \xrightarrow{\rho_N} P\tilde{\Gamma}_N \curvearrowright \mathbb{P}(\mathbb{R}^4) \\ \{a, b, c\} \mapsto \{A, B, C\} \end{gathered} \tag{3.1.4}$$

which will be our basic object of study.

3.2. The cones

The action of the reflections b, c on the boundary of hyperbolic space divides it into $2N$ circular arcs. We will associate a projective cone in $\mathbb{P}(\mathbb{R}^4)$ to one of the arcs, and propagate it to the remaining $2N-1$ arcs using the action of the matrices B, C.

3.2.1. Cones and projective cones. To specify a cone in $\mathbb{R}^4$ one can either give the vectors spanning it, or specify the equations of its faces. This, in particular, gives a duality between cones in a vector space and cones in its dual. Our cones will turn out to be self-dual when the vector space is identified with its dual via the symplectic pairing. Additional, our cones will be simplicial, i.e. have four faces and four extreme rays.

We will describe a (simplicial) cone $\mathcal{C}$ by specifying its four spanning vectors, and write $\mathcal{C} = [v_0|v_1|v_2|v_3]$ where v_i are column vectors in $\mathbb{R}^4$. So elements of $\mathcal{C}$ are of the form $\sum a_i v_i$ where $a_i \geq 0$. For the ping-pong argument we will consider the image of the cones in $\mathbb{P}(\mathbb{R}^4)$, but for calculations we will distinguish between a cone $\mathcal{C}$ and its negative $-\mathcal{C}$, spanned by $-v_i$. Given a cone $\mathcal{C}$ in $\mathbb{R}^4$ we denote by $\mathbb{P}\mathcal{C}$ its image in $\mathbb{P}(\mathbb{R}^4)$.

3.2.2. The ping-pong cones. Let us postpone the explicit definition of the cone vectors until §3.4 but use the following notation to describe some important properties. We start with a cone of the form

$$\mathcal{C}_0 := [v_0 \,|\, v_1 \,|\, v_2 \,|\, v_3] \tag{3.2.3}$$

where $v_i \in \mathbb{R}^4$ are vectors with the following properties:

(i) The vectors v_0, v_2 are fixed by B.
(ii) The vector v_3 satisfies $Cv_3 = -v_3$.

In particular $Bv_3 = B(-Cv_3) = (-R)v_3$.

We can then define the adjacent cone by reflection in B:

$$\mathcal{C}'_0 := B\mathcal{C}_0 = [v_0 \,|\, Bv_1 \,|\, v_2 \,|\, (-R)v_3] \tag{3.2.4}$$

All the other cones are obtained by applying the rotation matrix to these basic cones, specifically:

$$\mathcal{C}_k := R^k\mathcal{C}_0 \quad \mathcal{C}'_k := R^k\mathcal{C}'_0 \quad k = 1, \dots, N-1.$$

Our main result can then be stated as follows:

3.2.5. Theorem (Ping-pong property of cones)**.** *Consider the projective cones $\mathbb{P}\mathcal{C}_k, \mathbb{P}\mathcal{C}'_k$ for $k = 0, \dots, N-1$ defined above.*

(i) *The interiors of distinct cones are disjoint.*
(ii) *For any of the cones $\mathbb{P}\mathcal{C}$ except $\mathcal{C}_0$, we have that $A \cdot \mathbb{P}\mathcal{C} \subset \mathbb{P}\mathcal{C}_0$ where A is the matrix from §2.2.4.*

Therefore

$$P\tilde{\Gamma}_N \cong \left\langle a, b, c \,|\, a^2 = b^2 = c^2 = 1, (bc)^N = 1 \right\rangle$$

as a group.

3.3. Proof of the ping-pong property

In this section we reduce the proof of Theorem 3.2.5 to certain positivity properties that will be verified in Section 4.

3.3.1. Disjointness property. The dihedral group $\langle B, C\rangle$ acts freely and transitively on the set of projective cones $\{\mathbb{P}\mathcal{C}_i, \mathbb{P}\mathcal{C}'_i\}$. Therefore, to verify disjointness of any pair, it suffices to verify disjointness of $\mathbb{P}\mathcal{C}_0$ from any other cone in the list. Furthermore, because we work projectively, it suffices to show that the cones $(-R)^k\mathcal{C}_0$ and $(-R)^k\mathcal{C}'_0$ are disjoint from $\mathcal{C}_0$ and $-\mathcal{C}_0$ in $\mathbb{R}^4$.

3.3.2. Contraction property. Continuing to make use of the freedom to work projectively, for contraction it suffices to verify that $(-1)^k\mathcal{C}_k$ is mapped by A into $\mathcal{C}_0$, and similarly for $(-1)^k\mathcal{C}'_k$. Recall that we have

$$\begin{aligned}(-1)^k\mathcal{C}_k &= (-R)^k\mathcal{C}_0 = (-R)^k\,[v_0\,|\,v_1\,|\,v_2\,|\,v_3]\\ (-1)^k\mathcal{C}'_k &= (-R)^k\mathcal{C}'_0 = (-R)^k\,[v_0\,|\,Bv_1\,|\,v_2\,|\,(-R)v_3]\end{aligned} \tag{3.3.3}$$

So for contraction it suffices to verify that the vectors

$$(-R)^k v_0 \quad (-R)^k v_1 \quad (-R)^k v_2 \quad (-R)^k v_3 \quad (-R)^k Bv_1 \tag{3.3.4}$$

are mapped by A into the (closed) original cone $\mathcal{C}_0$, for any $k = 1,\ldots,N-1$. Similarly, for disjointness of cones it suffices to verify that the above vectors are themselves disjoint from the (closed) cone and its opposite (recall that we must check disjointness projectively). Let us emphasize here that we will verify this assertion for the cone $\mathcal{C}_0$ and vectors viewed in $\mathbb{R}^4$, not their projective versions.

Note that there is one exceptional case, namely for the vector v_3 we also have to consider $(-R)^N v_3$. However $(-R)^N = -\mathbf{1}$ independently of the parity of N, and $Av_3 = -v_3$, so the required positivity properties will follow straightforwardly in this endpoint case.

3.3.5. Verifying inclusion in a cone. For a simplicial cone $\mathcal{C}$ in $\mathbb{R}^4$, we will use the same letter for the matrix of its columns. In general, to certify that a column vector v belongs to $\mathcal{C}$ one must first compute a matrix $\check{\mathcal{C}}$, which defines the faces of $\mathcal{C}$, and check that $\check{\mathcal{C}}v$ has only non-negative entries. Conversely, if the result has at least one strictly negative entry, the vector is not in the cone.

Our cone $\mathcal{C}_0$ will have an additional self-duality property under the symplectic pairing. Specifically, we will find that $\mathcal{C}_0^t \cdot J \cdot \mathcal{C}_0$ is anti-diagonal, where J is the matrix of the symplectic pairing. Let now S be a diagonal matrix with the same signs on the diagonal as the anti-diagonal matrix $\mathcal{C}_0^t \cdot J \cdot \mathcal{C}_0$. We are thus lead to define the matrix:

$$M := S \cdot \mathcal{C}_0^t \cdot J \tag{3.3.6}$$

which gives the following certificate on a column vector $v \in \mathbb{R}^4$. Consider Mv: if all entries are non-negative then v belongs to $\mathcal{C}_0$, and if at least one entry is strictly negative then it is in the exterior. Furthermore, if one entry is strictly negative and one is strictly positive, then v is disjoint both from $\mathcal{C}$ and $-\mathcal{C}$, so disjoint projectively.

3.3.7. Contraction implies disjointness. By the discussion in the preceding paragraphs, our task is reduced to showing that for certain vectors v listed in Eqn. (3.3.4), the vector MAv has all entries positive (to certify contraction) while Mv has two entries of opposite sign (to certify projective disjointness).

Our matrix M will have the further useful property that its rows are eigenvectors of A, with eigenvalues ± 1. Specifically the first and third rows are (right) eigenvectors with eigenvalue $+1$, and the second and fourth rows have eigenvalue -1. So if all entries of MAv are non-negative, and at least three entries are not zero, then Mv has two entries of opposite sign.

Note that the property of M to have rows which are (right) eigenvectors of A is equivalent, by the construction of M, to the property that the original cone $\mathcal{C}_0$ is spanned by eigenvectors of A, with the sign pattern of eigenvalues flipped since $A^t J A = -J$.

3.3.8. Summary. To sum up, we have reduced the proof of Theorem 3.2.5 to showing that vectors of the form $MA(-R)^k v$ have all entries non-negative (and some strictly positive) for $k = 1 \dots N-1$ and $v \in \{v_0, v_1, v_2, v_3, Bv_1\}$. We next exhibit in §3.4 the explicit vectors and matrices described above and verify that they satisfy the useful properties we stated. We then proceed to actually verify the required positivity properties in Section 4.

3.4. Explicit cones and properties

3.4.1. Further abbreviations. Besides the abbreviations c_i, s_i for cosines and sines from §2.2.1, the following quantities

$$cc_1 := 1 - c_1 \qquad\qquad cc_2 := 1 - c_2 \tag{3.4.2}$$

will prove useful. Additionally it will prove useful to introduce the quantities:

$$\begin{aligned} L_1 &:= \frac{cc_1 \cdot cc_2 - 3(cc_2 - cc_1)}{-cc_1 \cdot cc_2 + 3(cc_1 + cc_2)} \\ L_2 &:= \frac{cc_1 \cdot cc_2 + 3(cc_2 - cc_1)}{-cc_1 \cdot cc_2 + 3(cc_1 + cc_2)} \end{aligned} \tag{3.4.3}$$

These quantities satisfy a number of useful identities which will be discussed below. The main ones, which characterize the L_i in terms of cc_i, are:

$$3(L_1 + L_2) = (L_1 + 1)cc_2 = (L_2 + 1)cc_2 \tag{3.4.4}$$

We will deduce some further properties of these quantities in §4.2.8

3.4.5. The cone. With these preparations, here is the cone:

$$\mathcal{C}_0 := \begin{bmatrix} 0 & -L_2 \cdot cc_1 & 0 & cc_1 \\ -L_1 \cdot \frac{cc_1}{s_1} & \frac{-cc_1^2}{s_1} & -\frac{cc_1}{s_1} & -\frac{cc_1^2}{s_1} \\ 0 & L_1 \cdot cc_2 & 0 & -cc_2 \\ L_2 \cdot \frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} & \frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} \end{bmatrix} \tag{3.4.6}$$

3.4.7. Self-duality of the cone. Recall that we introduced the matrix of the symplectic pairing J in Eqn. (2.2.5). Then it is a direct algebraic verification that:

$$\mathcal{C}_0^t \cdot J \cdot \mathcal{C}_0 = \begin{bmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{bmatrix} \qquad \alpha := 2 \cdot \frac{L_1 - L_2}{cc_1 - cc_2} \cdot \frac{cc_1^2 cc_2^2}{s_1 s_2}$$

The verification of this is immediate, the only non-trivial algebraic manipulation is to show that the fourth entry in the second row, and the second entry in the fourth row, vanish. This ultimately follows from the identity $(L_1+1)cc_2 = (L_2+1)cc_1$ stated in Eqn. (3.4.4). It is important to note that $\alpha > 0$, since $L_2 > L_1$ and $cc_1 > cc_2$, as will be established later in §4.2.8.

3.4.8. Contraction matrix. The identity for $\mathcal{C}_0^t J \mathcal{C}_0$ from the previous paragraph implies that the equations for the faces of the cone are given by taking the symplectic pairing against the spanning vectors (with appropriate signs). So if we denote by S' the diagonal matrix with entries $1, -1, 1, -1$ it follows that $S' \cdot \mathcal{C}_0^t \cdot J$ is the matrix that detects if a vector belongs, or not, to the cone $\mathcal{C}_0$. In fact, we could have used instead of S' any diagonal matrix with entries having the same pattern of signs. Let us therefore consider the matrix :

$$M := \tfrac{1}{\alpha'} \cdot S' \cdot \mathcal{C}_0^t \cdot J \quad \text{where} \quad \alpha' := \frac{2}{cc_2 - cc_1} \cdot \frac{cc_1^2 cc_2^2}{s_1 s_2} > 0$$

Computing it explicitly yields:

$$M = \begin{bmatrix} -\frac{L_1}{cc_1} & 0 & -\frac{L_2}{cc_2} & 0 \\ 1 & -L_2 \frac{s_1}{cc_1} & 1 & -L_1 \frac{s_2}{cc_2} \\ -\frac{1}{cc_1} & 0 & -\frac{1}{cc_2} & 0 \\ 1 & \frac{s_1}{cc_1} & 1 & \frac{s_2}{cc_2} \end{bmatrix} \tag{3.4.9}$$

3.4.10. Why contraction implies disjointness. Recall that in order to establish Theorem 3.2.5, we must verify that for certain vectors v the vector MAv has all entries non-negative (the contraction action of A) and that for the same vectors Mv has two entries of opposite sign.

We next observe that the matrices M and MA have the property that their second and fourth rows agree, while the first and third are negative of each other. Specifically, a direct computation yields:

$$MA = \begin{bmatrix} \frac{L_1}{cc_1} & 0 & \frac{L_2}{cc_2} & 0 \\ 1 & -L_2\frac{s_1}{cc_1} & 1 & -L_1\frac{s_2}{cc_2} \\ \frac{1}{cc_1} & 0 & \frac{1}{cc_2} & 0 \\ 1 & \frac{s_1}{cc_1} & 1 & \frac{s_2}{cc_2} \end{bmatrix} \tag{3.4.11}$$

In order to verify this identity, we just multiply the expression for M from Eqn. (3.3.6) and for A from §2.2.4. The only identity that needs to be used (when computing the first and fourth entries of the second row) is that

$$1 = \frac{L_1 cc_2 - L_2 cc_1}{cc_1 - cc_2}$$

which follows readily from Eqn. (3.4.4).

From the formulas for the matrices M and MA, it is now clear that if MAv has all entries non-negative, and at least one entry in each of the pairs {first,third} and {second,fourth} is nonzero, then automatically Mv will have two entries of opposite sign. It follows that it suffices to consider vectors of the form MAv and establish that their entries are non-negative; it will be transparent from the calculations that the needed nonvanishing will also hold.

4. Computations with rotated vectors

Outline of section. In §4.1 we introduce powers of the rotation matrix which is used to transport the vectors of interest. Next we introduce some further notation and some background calculations in §4.2.

The bulk of the calculations is performed in the remaining sections. We tackle the vectors in increasing order of complexity, namely v_2, v_0, v_3, v_1, Bv_1.

4.1. The vectors, their rotations, and contraction

Recall that the vectors we need to consider are v_0, v_1, v_2, v_3 and Bv_1, where the v_i span the cone $\mathcal{C}_0$. Here are the vectors:

$$[v_0|v_1|v_2|v_3|Bv_1] = \begin{bmatrix} 0 & -L_2 cc_1 & 0 & cc_1 & -L_2 cc_1 \\ -L_1\frac{cc_1}{s_1} & \frac{-cc_1^2}{s_1} & -\frac{cc_1}{s_1} & -\frac{cc_1^2}{s_1} & \frac{cc_1^2}{s_1} \\ 0 & L_1 cc_2 & 0 & -cc_2 & L_1 cc_2 \\ L_2\frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} & \frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} & \frac{-cc_2^2}{s_2} \end{bmatrix} \tag{4.1.1}$$

4.1.2. Passing to θ_N-parameter. When facing trigonometric expressions, it will be convenient to express everything in terms of the basic angle

$$\theta_N = \frac{\pi}{N} \text{ so that } \mu_1 = \pi - 3\theta_N \text{ and } \mu_2 = \pi - \theta_N. \tag{4.1.3}$$

It is then immediate that:

$$\begin{aligned} \cos(\mu_1) &= \cos(\pi - 3\theta_N) = -\cos(3\theta_N) \\ \sin(\mu_1) &= \sin(\pi - 3\theta_N) = \sin(3\theta_N) \\ \cos(\mu_2) &= \cos(\pi - \theta_N) = -\cos(\theta_N) \\ \sin(\mu_2) &= \sin(\pi - \theta_N) = \sin(\theta_N) \end{aligned} \tag{4.1.4}$$

Recall next that the matrix $R = BC$ is block-diagonal, rotating in the first block by μ_1 and in the second by μ_2. Therefore we have:

$$(-R) = \begin{bmatrix} \cos(3\theta_N) & \sin(3\theta_N) & 0 & 0 \\ -\sin(3\theta_N) & \cos(3\theta_N) & 0 & 0 \\ 0 & 0 & \cos(\theta_N) & \sin(\theta_N) \\ 0 & 0 & -\sin(\theta_N) & \cos(\theta_N) \end{bmatrix}$$

so $(-R)$ is rotation by $-3\theta_N$ in the first block and by $-\theta_N$ in the second block. To abbreviate further the sines of multiple angles, we will use the notation:

$$\begin{aligned} cp_1 &:= \cos(3k\theta_N) & \qquad sp_1 &:= -\sin(3k\theta_N) \\ cp_2 &:= \cos(k\theta_N) & \qquad sp_2 &:= -\sin(k\theta_N) \end{aligned} \tag{4.1.5}$$

where *cp* and *sp* are meant to denote "cosine power" and "sine power". These are precisely the entries of $(-R)^k$:

$$(-R)^k = \begin{bmatrix} cp_1 & -sp_1 & 0 & 0 \\ sp_1 & cp_1 & 0 & 0 \\ 0 & 0 & cp_2 & -sp_2 \\ 0 & 0 & sp_2 & cp_2 \end{bmatrix} \tag{4.1.6}$$

We have chosen to express $(-R)^k$ with the sign choices standard for a counterclockwise rotation matrix, but the reader should keep in mind that the signs of sines are as stated in Eqn. (4.1.5).

Let us finally recall the matrix MA from Eqn. (3.4.11):

$$MA = \begin{bmatrix} \frac{L_1}{cc_1} & 0 & \frac{L_2}{cc_2} & 0 \\ 1 & -L_2\frac{s_1}{cc_1} & 1 & -L_1\frac{s_2}{cc_2} \\ \frac{1}{cc_1} & 0 & \frac{1}{cc_2} & 0 \\ 1 & \frac{s_1}{cc_1} & 1 & \frac{s_2}{cc_2} \end{bmatrix} \tag{4.1.7}$$

Our task has been reduced to computing $MA(-R)^k v$, for $k = 1 \dots N-1$, and for each column vector v in Eqn. (4.1.1).

This will take up the rest of this section, after some preliminary recollections from trigonometry in the next section.

4.2. Frequently used expressions

4.2.1. Some trigonometric formulas. Since $\mu_1 \equiv 3\mu_2 \mod 2\pi$ it will be useful to make use of angle-tripling formulas:

$$\begin{aligned} \sin(3\theta) &= \sin(\theta) \cdot \left(3 - 4\sin^2(\theta)\right) = \sin(\theta) \cdot (2\cos(2\theta) + 1) \\ \cos(3\theta) &= \cos(\theta) \cdot \left(4\cos^2(\theta) - 3\right) = \cos(\theta) \cdot (2\cos(2\theta) - 1) \end{aligned} \tag{4.2.2}$$

Besides considering $\cos(3\theta)/\cos(\theta)$ and similarly for sine, we will frequently also use the following difference of cosines:

$$\begin{aligned} \cos(3\theta) - \cos(\theta) &= \cos(\theta) \cdot 2 \cdot (\cos(2\theta) - 1) \\ &= -4 \cdot \cos(\theta) \cdot \sin^2(\theta) \\ &= -\sin(\theta) \cdot 2 \cdot \sin(2\theta) \end{aligned} \tag{4.2.3}$$

and its analogue for sines:

$$\sin(3\theta) - \sin(\theta) = \sin(\theta) \cdot 2 \cdot \cos(2\theta). \tag{4.2.4}$$

We'll also make use of the standard addition/subtraction formulas:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ \sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \end{aligned} \tag{4.2.5}$$

4.2.6. Abbreviations. Recall that we introduced the algebraic expressions

$$\begin{aligned} cc_1 &:= 1 - c_1 \quad \text{so that} \quad \frac{cc_1}{s_1} = \frac{\sin(\mu_1/2)}{\cos(\mu_1/2)} = \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \\ cc_2 &:= 1 - c_2 \quad \text{so that} \quad \frac{cc_2}{s_2} = \frac{\sin(\mu_2/2)}{\cos(\mu_2/2)} = \frac{\cos(\theta_N/2)}{\sin(\theta_N/2)} \end{aligned} \tag{4.2.7}$$

where we used the formulas $1 - \cos(\theta) = 2\sin^2(\theta/2)$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$.

4.2.8. The parameters L_i. To shorten notation, we introduced in Eqn. (3.4.3) the constants L_1, L_2 which we will mostly use through the identities:

$$3(L_1 + L_2) = (L_1 + 1)cc_2 = (L_2 + 1)cc_1 = \tfrac{6 \cdot cc_1 \cdot cc_2}{-cc_1 \cdot cc_2 + 3(cc_1 + cc_2)} \tag{4.2.9}$$

which are verified directly from the definitions.

Let us record the basic inequalities which we use frequently :

$$cc_2 > cc_1 \quad \text{and} \quad L_2 > L_1$$

The first one is verified from the definitions of μ_i while the second one follows from the first, which we will also frequently write as $cc_2 - cc_1 > 0$. The relations between L_1 and L_2 imply

$$\begin{aligned} (L_1 + 1)cc_2 = 3(L_1 + L_2) > 6L_1 &\iff cc_2 > (6 - cc_2)L_1 \\ &\iff L_1 < \frac{cc_2}{6 - cc_2}. \end{aligned}$$

As $cc_2 < 2$ we obtain $L_1 < \frac{1}{2}$. This also implies the following

$$3(L_1 + L_2) = (L_1 + 1)cc_2 < \frac{3}{2} \cdot 2 = 3.$$

Hence

$$L_1 + L_2 < 1. \tag{4.2.10}$$

4.2.11. The difference of cosines. Using the triple-angle formula we find:

$$\begin{aligned} c_1 - c_2 = \cos(3\mu_2) - \cos(\mu_2) &= 4\cos(\mu_2)\Big[\cos(\mu_2)^2 - 1\Big] \\ &= -4c_2 s_2^2 \\ &= 4(c_2 - 1)c_2(c_2 + 1) \\ &> 0 \end{aligned}$$

The sign $c_1 - c_2 > 0$ follows from the above algebraic expressions and $c_2 < 0$, or by looking at the explicit values of μ_i.

4.2.12. The smallest and largest sines and cosines. Our rotation will range over $k = 1, \ldots, N - 1$. We have the elementary inequalities

$$0 < \sin(\mu_2) \leq \left|\sin\left(k\frac{N-1}{N}\pi\right)\right| \leq |\cos(\mu_2)| < 1$$

4.2.13. Frequently occurring differences of ratios. The following manipulation is used frequently, so we record it once here, using the properties of the ratios cc_i/s_i from Eqn. (4.2.7) as well as the addition formula for sines Eqn. (4.2.5):

$$\begin{aligned}\frac{cc_2}{s_2}-\frac{cc_1}{s_1} &= \frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\\ &= \frac{\sin(\theta_N)}{\sin(\theta_N/2)\sin(3\theta_N/2)}\\ &= \frac{2\cancel{\sin(\theta_N/2)}\cos(\theta_N/2)}{\cancel{\sin(\theta_N/2)}\sin(3\theta_N/2)}\\ &= 2\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}\end{aligned} \tag{4.2.14}$$

We now give a list of useful inequalities appearing with powers of the rotation.

$$\begin{aligned}\frac{sp_1}{s_1}-\frac{sp_2}{s_2} &= \frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot\left(\frac{-\sin(3k\theta_N)}{\sin(k\theta_N)}+\frac{\sin(3\theta_N)}{\sin(\theta_N)}\right)\\ &= 2\cdot\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot(\cos(2\theta_N)-\cos(2k\theta_N))\end{aligned} \tag{4.2.15}$$

In any event, this expression is always negative (non-positive) for $k = 1,\dots,N-1$. It vanishes precisely for $k = 1, N-1$.

Similarly

$$\begin{aligned}\frac{sp_1}{s_1}+\frac{sp_2}{s_2} &= -\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot\left(\frac{\sin(3k\theta_N)}{\sin(k\theta_N)}+\frac{\sin(3\theta_N)}{\sin(\theta_N)}\right)\\ &= -2\cdot\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot(\cos(2k\theta_N)+\cos(2\theta_N)+1)\end{aligned} \tag{4.2.16}$$

which is always non-positive.

Next we have the combinations

$$\frac{cc_1}{s_1}sp_1 \pm \frac{cc_2}{s_2}sp_2.$$

Recall that $cc_i = 1 - c_i$ and $c_1 = -\cos(3\theta_N), c_2 = -\cos(\theta_N)$ and we'll use the basic identity

$$1+\cos(\theta) = 2-2\sin^2(\theta/2) = 2\cos^2(\theta/2)$$

to reduce to the consideration of

(4.2.17)
$$\begin{aligned}\frac{cc_1}{s_1}sp_1-\frac{cc_2}{s_2}sp_2&=-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cdot\sin(3k\theta_N)+\frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}\cdot\sin(k\theta_N)\\&=\sin(k\theta_N)\left[-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cdot(2\cos(2k\theta_N)+1)+\frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}\right]\\&=\sin(k\theta_N)\left[-2\cos(2k\theta_N)\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}+\frac{\sin(\theta_N)}{\sin(3\theta_N/2)\sin(\theta_N/2)}\right]\\&=\frac{2\sin(k\theta_N)}{\sin(3\theta_N/2)}\left[\cos\left(\frac{\theta_N}{2}\right)-\cos(2k\theta_N)\cos\left(\frac{3\theta_N}{2}\right)\right]\end{aligned}$$

This expression is manifestly non-negative, since the value $\cos(\theta_N/2)$ is larger that $\cos(3\theta_N/2)$ for all $N\geq 4$.

Let's do the same but with a sum:

(4.2.18)
$$\begin{aligned}\frac{cc_1}{s_1}sp_1+\frac{cc_2}{s_2}sp_2&=-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cdot\sin(3k\theta_N)-\frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}\cdot\sin(k\theta_N)\\&=\sin(k\theta_N)\left[-2\cos(2k\theta_N)\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}-\frac{\sin(2\theta_N)}{\sin(3\theta_N/2)\sin(\theta_N/2)}\right]\\&=-\frac{2\sin(k\theta_N)}{\sin(3\theta_N/2)}\Big[\cos(2k\theta_N)\cos(3\theta_N/2)+2\cos(\theta_N)\cos(\theta_N/2)\Big]\end{aligned}$$

This is non-positive, in fact the term $\cos(\theta_N)\cos(\theta_N/2)$ is larger in absolute value than the other one $\cos(2k\theta_N)(\cos(3\theta_N/2))$.

We give in the following two propositions simplified expression for some combinations of terms that appear several times in our computations.

4.2.19. Proposition. *The following formulas hold:*

$$\begin{aligned}\pm cp_1-\frac{cc_1}{s_1}sp_1&=\frac{\sin(3(k\mp\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\\\pm cp_2-\frac{cc_2}{s_2}sp_2&=\frac{\sin((k\mp\frac{1}{2})\theta_N)}{\sin(\theta_N/2)}\\\pm(cp_1-cp_2)-\frac{cc_1}{s_1}sp_1+\frac{cc_2}{s_2}sp_2&=-4\cdot\frac{\sin((k\pm\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\cdot\sin\left(k\theta_N\right)\cdot\sin\left((k\pm1)\theta_N\right)\end{aligned}$$

Proof. On one hand we have

$$\begin{aligned}\pm cp_1 - \frac{cc_1}{s_1} sp_1 &= \mp \cos(3k\theta_N) + \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \sin(3k\theta_N) \\ &= \frac{\sin(3(k \mp \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\end{aligned}$$

and similarly for the second equality.

We now consider:

$$\begin{aligned}\frac{\sin(3(k \mp \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} - \frac{\sin((k \mp \frac{1}{2})\theta_N)}{\sin(\theta_N/2)} &= \frac{\sin((k \mp \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \left(\frac{\sin(3(k \mp \frac{1}{2})\theta_N)}{\sin((k \mp \frac{1}{2})\theta_N)} - \frac{\sin(3\theta_N/2)}{\sin(\theta_N/2)} \right) \\ &= 2 \cdot \frac{\sin((k \mp \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \left(\cos((2k \mp 1)\theta_N) - \cos(\theta_N)\right) \\ &= -4 \cdot \frac{\sin((k \mp \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k \mp 1)\theta_N)\end{aligned}$$

□

The second proposition is for a similar expression where we invert the fractions.

4.2.20. Proposition. *The following formulas hold:*

$$\begin{aligned}cp_1 \pm \frac{s_1}{cc_1} sp_1 &= \frac{\cos(3(k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \\ cp_2 \pm \frac{s_2}{cc_2} sp_2 &= \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(\theta_N/2)} \\ cp_1 - cp_2 \pm \left(\frac{s_1}{cc_1} sp_1 - \frac{s_2}{cc_2} sp_2 \right) &= -4 \cdot \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k \pm 1)\theta_N)\end{aligned}$$

Proof. On one hand we have

$$\begin{aligned}cp_1 \pm \frac{s_1}{cc_1} sp_1 &= \cos(3k\theta_N) \mp \frac{\sin(3\theta_N/2)}{\cos(3\theta_N/2)} \sin(3k\theta_N) \\ &= \frac{\cos(3(k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}\end{aligned}$$

and similarly for the second equality.

We now consider

$$\frac{\cos(3(k\pm\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} - \frac{\cos((k\pm\frac{1}{2})\theta_N)}{\cos(\theta_N/2)} = \frac{\cos((k\pm\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}\left(\frac{\cos(3(k\pm\frac{1}{2})\theta_N)}{\cos((k\pm\frac{1}{2})\theta_N)} - \frac{\cos(3\theta_N/2)}{\cos(\theta_N/2)}\right)$$
$$= 2\cdot\frac{\cos((k\pm\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}\left(\cos((2k\pm 1)\theta_N) - \cos(\theta_N)\right)$$
$$= -4\cdot\frac{\cos((k\pm\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}\cdot\sin\left(k\theta_N\right)\cdot\sin\left((k\pm 1)\theta_N\right)$$

$\square$

We are now ready to proceed to the analysis of vectors.

4.3. Computing with v_2

The vector v_2 is the third column of the matrix M:

$$v_2 = \begin{bmatrix} 0 \\ -\frac{cc_1}{s_1} \\ 0 \\ \frac{cc_2}{s_2} \end{bmatrix} \text{ so } (-R)^k v_2 = \begin{bmatrix} \frac{cc_1}{s_1} sp_1 \\ -\frac{cc_1}{s_1} cp_1 \\ -\frac{cc_2}{s_2} sp_2 \\ \frac{cc_2}{s_2} cp_2 \end{bmatrix}$$

Applying the matrix MA yields:

$$MA(-R)^k v_2 = \begin{bmatrix} L_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot \frac{sp_2}{s_2} \\ L_2 \cdot cp_1 - L_1 \cdot cp_2 + cc_1 \cdot \frac{sp_1}{s_1} - cc_2 \cdot \frac{sp_2}{s_2} \\ \frac{sp_1}{s_1} - \frac{sp_2}{s_2} \\ -cp_1 + cp_2 + cc_1 \cdot \frac{sp_1}{s_1} - cc_2 \cdot \frac{sp_2}{s_2} \end{bmatrix}$$

4.3.1. The third entry of $MA(-R)^k v_2$. The third entry of the vector is:

$$\frac{sp_1}{s_1} - \frac{sp_2}{s_2} = \frac{-\sin(3k\theta_N)}{\sin(3\theta_N)} - \frac{-\sin(k\theta_N)}{\sin(\theta_N)}$$

We rewrite this as:

$$\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\left(\frac{\sin(3\theta_N)}{\sin(\theta_N)} - \frac{\sin(3k\theta_N)}{\sin(k\theta_N)}\right)$$

The first factor is clearly positive for our range of k, and we rewrite the difference using the formula for sine of the triple angle (Eqn. (4.2.2)) as:

$$\left(\frac{\sin(3\theta_N)}{\sin(\theta_N)} - \frac{\sin(3k\theta_N)}{\sin(k\theta_N)}\right) = 2\cos(2\theta_N) - 2\cos(2k\theta_N) \geq 0$$

and equality holds precisely for $k = 1, N-1$.

4.3.2. The fourth entry of $MA(-R)^k v_2$. By Proposition 4.2.19, the 4th entry of the vector is

$$-cp_1+cp_2+\frac{cc_1}{s_1}sp_1-\frac{cc_2}{s_2}sp_2 = 4\cdot\frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\cdot\sin(k\theta_N)\cdot\sin((k+1)\theta_N)$$

which is non-negative for $1 \leq k \leq N-1$.

4.3.3. The second entry of $MA(-R)^k v_2$. We have to show that

$$L_2 cp_1 - L_1 cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} \geq 0$$

Observing the similarity to the fourth entry, we divide the claim into two parts. For $k \geq \frac{N}{2}$ we claim that the above quantity is clearly greater or equal to the fourth entry, hence non-negative. For $\frac{N}{2} \geq k$, we will replace $L_2cp_1 - L_1cp_2$ by $cp_1 - cp_2$, as detailed in §4.3.6 below, and proceed as with the fourth entry.

4.3.4. The case $N > k \geq N/2$. To check that the second entry is greater than the fourth entry reduces to the inequality

$$(L_2+1)cp_1 - (L_1+1)cp_2 \geq 0$$

We now rewrite this, taking into account the identities for L_i from Eqn. (4.2.9):

$$(L_2+1)cp_1-(L_1+1)cp_2 = (L_2+1)cc_1\frac{cp_1}{cc_1}-(L_1+1)cc_2\frac{cp_2}{cc_2} = 3\,(L_1+L_2)\left(\frac{cp_1}{cc_1}-\frac{cp_2}{cc_2}\right)$$

The factor $3(L_1+L_2)$ is positive, so we can drop it. We can also factor out $\frac{-cp_2}{cc_1}$, taking into account that $cp_2 = \cos(k\theta_N) \leq 0$ when $N \geq k \geq \frac{N}{2}$, to reduce to showing:

$$\frac{cc_1}{cc_2} - \frac{cp_1}{cp_2} \geq 0 \tag{4.3.5}$$

Now we rewrite everything in terms of θ_N and use the angle-tripling formula for cosines to express the terms as:

$$\frac{cc_1}{cc_2} = \frac{1+\cos(3\theta_N)}{1+\cos(\theta_N)} \quad \frac{cp_1}{cp_2} = 2\cos(2k\theta_N)-1$$

We next reduce the expressions:

$$\frac{1+\cos(3\theta_N)}{1+\cos(\theta_N)} \geq 2\cos(2k\theta_N)-1 \iff$$
$$\iff 1+\cos(3\theta_N) \geq 2\cos(2k\theta_N)-1+2\cos(\theta_N)\cos(2k\theta_N)-\cos(\theta_N)$$
$$\iff 1+\frac{\cos(3\theta_N)+\cos(\theta_N)}{2} \geq \cos(2k\theta_N)+\cos(\theta_N)\cos(2k\theta_N)$$

The left-hand side above is independent of k while the right-hand side is monotonically increasing and achieves its maximum when $k = N-1$ to reduce to

$$1 + \frac{\cos(3\theta_N) + \cos(\theta_N)}{2} \geq \cos(2\theta_N) + \cos(\theta_N)\cos(2\theta_N)$$

Finally we use again the angle-tripling formula to find $\cos(3\theta_N) + \cos(\theta_N) = 2\cos(\theta_N)\cos(2\theta_N)$ and reduce to

$$1 + \cancel{\cos(\theta_N)\cos(2\theta_N)} \geq \cos(2\theta_N) + \cancel{\cos(\theta_N)\cos(2\theta_N)}$$

which clearly holds.

4.3.6. The case $N/2 \geq k \geq 1$. In this case we claim that we have

$$L_2 cp_1 - L_1 cp_2 \geq cp_1 - cp_2.$$

Assuming this for the moment, we proceed as in the analysis of the fourth entry but this time subtracting $cp_2 - cp_1$, i.e.

$$L_2 cp_1 - L_1 cp_2 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2} \geq cp_1 - cp_2 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2}.$$

This last expression can be rewritten using Proposition 4.2.19

$$4 \cdot \frac{\sin((k - \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k-1)\theta_N).$$

Which is positive for $1 \leq k \leq N-1$.

4.3.7. The first entry of $MA(-R)^k v_2$. We have to show that the expression

$$L_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot \frac{sp_2}{s_2}$$

is non-negative. We factor out the term $-sp_2/s_1$ and it suffices to show that the resulting expression

$$L_2 \frac{s_1}{s_2} - L_1 \cdot \frac{sp_1}{sp_2}$$

is positive since $s_1 > 0$ and $-sp_2 = \sin(k\theta_N) > 0$. Both ratios sp_1/sp_2 and s_1/s_2 are ratios of a triple sine over a sine, so we rewrite them using Eqn. (4.2.2) to find :

$$L_2(3 - 4s_2^2) - L_1(3 - 4sp_2^2)$$

We rewrite this, using a “polarization” identity:

$$Ax - By = \frac{A-B}{2}(x+y) + \frac{A+B}{2}(x-y). \tag{4.3.8}$$

After multiplying by 2, we get

$$(L_2 - L_1)\left[6 - 4(s_2^2 + sp_2^2)\right] + (L_2 + L_1)\left[sp_2^2 - s_2^2\right].$$

Now $L_2 - L_1 > 0$ and $s_2^2 = \sin^2(\theta_N) \leq \frac{1}{2}$ since $N \geq 3$, so the first term is clearly positive. For the second one we observe that $sp_2^2 \geq s_2^2$ since this is saying that $\sin(k\theta_N) \geq \sin(\theta_N)$ for $k = 1, \dots N-1$.

4.4. Computing with v_0

4.4.1. Setup. Recall that

$$v_0 = \begin{bmatrix} 0 \\ -L_1 \frac{cc_1}{s_1} \\ 0 \\ L_2 \frac{cc_2}{s_2} \end{bmatrix} \text{ so } (-R)^k v_0 = \begin{bmatrix} L_1 \frac{cc_1}{s_1} sp_1 \\ -L_1 \frac{cc_1}{s_1} cp_1 \\ -L_2 \frac{cc_2}{s_2} sp_2 \\ L_2 \frac{cc_2}{s_2} cp_2 \end{bmatrix}.$$

So the vector that we have to analyze, namely $MA(-R)^k v_0$ is:

$$\begin{bmatrix} L_1^2 \cdot \frac{sp_1}{s_1} - L_2^2 \cdot \frac{sp_2}{s_2} \\ L_1 L_2 \cdot cp_1 - L_1 L_2 \cdot cp_2 + L_1 \cdot cc_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot cc_2 \cdot \frac{sp_2}{s_2} \\ L_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot \frac{sp_2}{s_2} \\ -L_1 \cdot cp_1 + L_2 \cdot cp_2 + L_1 \cdot cc_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot cc_2 \cdot \frac{sp_2}{s_2} \end{bmatrix} \tag{4.4.2}$$

4.4.3. The third entry of $MA(-R)^k v_0$. Notice that the third entry is the same as the first entry of v_2, which we already checked is positive in §4.3.7.

4.4.4. The first entry of $MA(-R)^k v_0$. The first entry is similar to the one of v_2 dealt with in §4.3.7, we use the same method to show positivity.

First factor out the positive term $-sp_2/s_1$ and use the angle tripling formula Eqn. (4.2.2) to get

$$L_2^2(3 - 4s_2^2) - L_1^2(3 - 4sp_2^2)$$

Using the polarization identity Eqn. (4.3.8) after multiplying by 2, we get

$$(L_2^2 - L_1^2)\left[6 - 4(s_2^2 + sp_2^2)\right] + (L_2^2 + L_1^2)\left[sp_2^2 - s_2^2.\right]$$

Now $L_2^2 - L_1^2 > 0$ and $s_2^2 = \sin^2(\theta_N) \leq \frac{1}{2}$ since $N \geq 3$, so the first term is positive. For the second one use again that $sp_2^2 \geq s_2^2$ for $k = 1, \dots N-1$.

4.4.5. The fourth entry of $MA(-R)^k v_0$. Consider first the terms with a factor of L_1:

$$\begin{aligned} -\left(cp_1 - cc_1 \frac{sp_1}{s_1}\right) &= -\left(\cos(3k\theta_N) + \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \sin(3k\theta_N)\right) \\ &= -\frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}. \end{aligned}$$

Similarly consider those with a factor of L_2:

$$cp_2 - cc_2\frac{sp_2}{s_2} = \frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)}.$$

The fourth entry has thus the following expression

$$L_2\frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)} - L_1\frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}$$

We factor out the positive term $\sin\left((k+\frac{1}{2})\theta_N\right)/\sin(3\theta_N/2)$,

$$\begin{aligned} L_2\frac{\sin(3\theta_N/2)}{\sin(\theta_N/2)} - L_1\frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin\left((k+\frac{1}{2})\theta_N\right)} - &= L_2\left(2\cos(\theta_N)+1\right) - L_1\left(2\cos\left((2k+1)\theta_N\right)+1\right) \\ &= (L_2 - L_1) + 2\left(L_2\cos(\theta_N) - L_1\cos\left((2k+1)\theta_N\right)\right). \end{aligned}$$

As $L_2 > L_1 > 0$ and $\cos(\theta_N) \geq \cos((2k+1)\theta_N)$ for all k this expression is positive.

4.4.6. The second entry of $MA(-R)^k v_0$. Observing the similarity to the fourth entry, we divide the claim into two parts. For $k \geq \frac{N}{2}$ we claim that the above quantity is greater than or equal to the fourth entry, hence non-negative. For $\frac{N}{2} \geq k$, we use relations with L_1, L_2.

4.4.7. The case $N > k \geq N/2$. To check that the second entry is greater than the fourth entry reduces to the inequality

$$L_1(L_2+1)cp_1 - L_2(L_1+1)cp_2 \geq 0$$

Using identities of L_i from Eqn. (4.2.9):

$$\begin{aligned} L_1(L_2+1)cp_1 - L_2(L_1+1)cp_2 &= (L_2+1)cc_1L_1\frac{cp_1}{cc_1} - (L_1+1)cc_2L_2\frac{cp_2}{cc_2} \\ &= 3\left(L_1+L_2\right)\left(L_1\frac{cp_1}{cc_1} - L_2\frac{cp_2}{cc_2}\right). \end{aligned}$$

The factors $3(L_1+L_2)$ and $\frac{-cp_2}{cc_1}$ are positive when $N \geq k \geq \frac{N}{2}$, then we are reduced to showing:

$$L_2\frac{cc_1}{cc_2} - L_1\frac{cp_1}{cp_2} \geq 0$$

As $L_2 > L_1 > 0$ this is implied by the inequality

$$\frac{cc_1}{cc_2} - \frac{cp_1}{cp_2} \geq 0$$

proved earlier, see Eqn. (4.3.5).

4.4.8. The case $N/2 \geq k \geq 1$. Notice that $L_2 > L_1$, $1-L_1 > 1-L_2$ always, and $cp_2 \geq cp_1$ in this range since $cp_2 - cp_1 = 2\sin(k\theta_N)\sin(2k\theta_N)$. Multiplying these three inequalities, we get

$$L_2(1-L_1)cp_2 \geq L_1(1-L_2)cp_1$$

then

$$L_1L_2cp_1 - L_1L_2cp_2 \geq L_1cp_1 - L_2cp_2.$$

Using this inequality we reduce the positivity of the second entry in this range to the positivity of

$$L_1(cp_1 + cc_1\frac{sp_1}{s_1}) - L_2(cp_2 + cc_2\frac{sp_2}{s_2}). \tag{4.4.9}$$

Using Proposition 4.2.19, we can rewrite this expression as:

$$L_2\frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin\left(\theta_N/2\right)} - L_1\frac{\sin\left(3(k-\frac{1}{2})\theta_N/2\right)}{\sin(3\theta_N/2)}.$$

Factoring out the positive term $\sin((k-\frac{1}{2})\theta_N)/\sin(3\theta_N/2)$,

$$\begin{aligned} L_2\frac{\sin\left(3\theta_N/2\right)}{\sin(\theta_N/2)} - L_1\frac{\sin\left(3(k-\frac{1}{2})\theta_N\right)}{\sin\left((k-\frac{1}{2})\theta_N\right)} &= L_2\left(2\cos(\theta_N)+1\right) - L_1\left(2\cos\left((2k-1)\theta_N\right)+1\right) \\ &= (L_2 - L_1) + 2\left(L_2\cos(\theta_N) - L_1\cos\left((2k-1)\theta_N\right)\right). \end{aligned}$$

As $L_2 > L_1 > 0$ and $\cos(\theta_N) \geq \cos((2k-1)\theta_N)$ for all k this expression is positive.

4.5. Computing with v_3

4.5.1. Setup. We have that

$$v_3 = \begin{bmatrix} cc_1 \\ -\frac{cc_1^2}{s_1} \\ -cc_2 \\ \frac{cc_2^2}{s_2} \end{bmatrix} \text{ so } (-R)^k v_3 = \begin{bmatrix} cc_1 \cdot cp_1 + \frac{cc_1^2}{s_1}\cdot sp_1 \\ -\frac{cc_1^2}{s_1}\cdot cp_1 + cc_1 \cdot sp_1 \\ -cc_2\cdot cp_2 - \frac{cc_2^2}{s_2}\cdot sp_2 \\ \frac{cc_2^2}{s_2}\cdot cp_2 - cc_2\cdot sp_2 \end{bmatrix}$$

We then proceed to compute $MA(-R)^k v_3$ to be:

$$\begin{bmatrix} L_1\cdot cc_1\cdot\frac{sp_1}{s_1} - L_2\cdot cc_2\cdot\frac{sp_2}{s_2} + L_1cp_1 - L_2cp_2 \\ 3\,(L_1+L_2)\,(cp_1 - cp_2) - (L_2+1)\,s_1sp_1 + (L_1+1)\,s_2sp_2 + 2\left(cc_1\cdot\frac{sp_1}{s_1} - cc_2\cdot\frac{sp_2}{s_2}\right) \\ cp_1 - cp_2 + cc_1\cdot\frac{sp_1}{s_1} - cc_2\cdot\frac{sp_2}{s_2} \\ 2\left(cc_1\cdot\frac{sp_1}{s_1} - cc_2\cdot\frac{sp_2}{s_2}\right) \end{bmatrix}$$

4.5.2. The third entry of $MA(-R)^k v_3$. The third entry

$$cp_1 - cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}$$

was already proved to be positive in §4.3.6, using Proposition 4.2.19.

4.5.3. The first entry of $MA(-R)^k v_3$. The first entry of v_3 is the same as Eqn. (4.4.9) proved to be positive in §4.4.8.

4.5.4. The fourth entry of $MA(-R)^k v_3$. Factoring out 2, we recognize the last two terms of the fourth entry of v_2. As in §4.3.2 we use the triple-angle formula for sines:

$$\begin{aligned}\frac{cc_1}{s_1}sp_1 - \frac{cc_2}{s_2}sp_2 &= (-sp_2)\left(\frac{cc_2}{s_2} - \frac{cc_1}{s_1}\frac{sp_1}{sp_2}\right)\\ &= (-sp_2)\left(\frac{cc_2}{s_2} - \frac{cc_1}{s_1} - 2\frac{cc_1}{s_1}\cos(2k\theta_N)\right)\end{aligned}$$

now using Eqn. (4.2.14),

$$\begin{aligned}&= 2(-sp_2)\left(\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} - \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cos(2k\theta_N)\right)\\ &= 2\sin(k\theta_N)\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}\Big(1 - \frac{\cos(3\theta_N/2)}{\cos(\theta_N/2)}\cos(2k\theta_N)\Big)\\ &= 2\sin(k\theta_N)\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}\Big(1 - (2\cos(\theta_N) - 1)\cos(2k\theta_N)\Big)\end{aligned}$$

For all $1 \leq k < N$, $2\sin(k\theta_N)\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} > 0$. Moreover, as $N > 3$, we have $0 < 2\cos(\theta_N) - 1 < 1$ hence $1 - (2\cos(\theta_N) - 1)\cos(2k\theta_N) > 0$.

4.5.5. The second entry of $MA(-R)^k v_3$. As for the second entry of the previous vectors we divide the claim into two parts. For $k \geq \frac{N}{2}$ we claim that the above quantity is greater than or equal to the fourth entry, hence non-negative. For $\frac{N}{2} \geq k$ we use other relations.

4.5.6. The case $N > k > N/2$. We show in this case that

$$3(L_1 + L_2)(cp_1 - cp_2) - (L_2 + 1)s_1 sp_1 + (L_1 + 1)s_2 sp_2 \geq 0.$$

Using Eqn. (4.2.9), we can factor out $3(L_1 + L_2)$ to get an equivalent inequality

$$cp_1 - cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2 \geq 0.$$

By Proposition 4.2.20, this latter expression is equal to

$$-4\cdot\frac{\cos((k-\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}\cdot\sin(k\theta_N)\cdot\sin((k-1)\theta_N).$$

As $k > \frac{N}{2}$, $\cos((k-\frac{1}{2})\theta_N) < 0$ and moreover $k < N$ then $\sin(k\theta_N)$ and $\sin((k-1)\theta_N)$ are non-negative, thus the inequality is satisfied.

4.5.7. The case $1 \le k \le N/2$. First we use the fact $3(L_1+L_2) \le 3$ established in Eqn. (4.2.10). Moreover for $1 \le k \le N/2$, $\cos((k-\frac{1}{2})\theta_N) > 0$ thus according to the previous formula

$$cp_1 - cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2 \le 0.$$

Hence the second entry in this case is not smaller than

$$3 \cdot \left(cp_1 - cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2\right) + 2 \cdot \left(cc_1 \cdot \frac{sp_1}{s_1} - cc_2 \cdot \frac{sp_2}{s_2}\right).$$

Using the trigonometric computations of Proposition 4.2.20 and Eqn. (4.2.17) we can rewrite this expression as

$$-12 \cdot \frac{\cos((k-\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k-1)\theta_N)$$
$$+4\sin(k\theta_N)\left(\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} - \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cos(2k\theta_N)\right).$$

Factoring out the positive term $4 \cdot \frac{\sin(k\theta_N)}{\cos(3\theta/2)\cdot\sin(3\theta/2)}$ we are reduced to show positivity for

$$\begin{aligned} &-3\sin(3\theta_N/2)\cdot\cos((k-\frac{1}{2})\theta_N)\cdot\sin((k-1)\theta_N) \\ &+\cos(3\theta_N/2)\cdot\Big(\cos(\theta_N/2) - \cos(3\theta_N/2)\cos(2k\theta_N)\Big) \\ &= -\frac{3}{2}\sin(3\theta_N/2)\cdot\left(\sin((2k-\frac{3}{2})\theta_N) - \sin(\theta_N/2)\right) \\ &+\cos(3\theta_N/2)\cdot\Big(\cos(\theta_N/2) - \cos(3\theta_N/2)\cos(2k\theta_N)\Big) \end{aligned}$$

Hence we are reduced to showing that

$$\begin{aligned} &\frac{3}{2}\sin(3\theta_N/2)\cdot\sin((2k-\frac{3}{2})\theta_N) + \cos(3\theta_N/2)^2\cos(2k\theta_N) \\ &\quad \le \frac{3}{2}\sin(3\theta_N/2)\cdot\sin(\theta_N/2) + \cos(3\theta_N/2)\cos(\theta_N/2). \end{aligned}$$

In the following we show that the expression

$$\frac{3}{2}\sin(3\theta_N/2)\cdot\sin((2k-\frac{3}{2})\theta_N) + \cos(3\theta_N/2)^2\cos(2k\theta_N)$$

takes its maximal value at $k=1$.

4.5.8. Proposition. *Let a, b, ϵ be three positive parameters. Then there exists a unique $x_M \in [0,\pi)$ such that the map*

$$f(x) = a\sin(x-\epsilon) + b\cos(x)$$

is extremal at x_M. *Moreover* $\tan(x_M) = \frac{a\cos(\epsilon)}{b - a\sin(\epsilon)}$ *and* $f(x_M - x) = f(x)$.

Proof. We compute the derivative

$$f'(x) = a\cos(x-\epsilon) - b\sin(x).$$

We look for zeros of this map,

$$\begin{aligned} f'(x) = 0 &\iff a\cos(x-\epsilon) = b\sin(x) \\ \iff a\cos(x)\cos(\epsilon) &= (b - a\sin(\epsilon))\sin(x) \\ &\iff \tan(x) = \frac{a\cos(\epsilon)}{b - a\sin(\epsilon)}. \end{aligned}$$

The symmetry property comes from the fact that for all a, b, ϵ there exists A, D such that $f(x) = A\cos(x - D)$. □

We apply the proposition in the setting where $a = \frac{3}{2}\sin(3\theta_N/2)$, $b = \cos(3\theta_N/2)^2$, $\epsilon = 3\theta_N/2$. Then

$$\frac{a\cos(\epsilon)}{b - a\sin(\epsilon)} = \frac{3\sin(\epsilon)\cos(\epsilon)}{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2}.$$

Notice that $\tan(2\epsilon) = \frac{2\sin(\epsilon)\cos(\epsilon)}{\cos(\epsilon)^2 - \sin^2(\epsilon)}$. Hence

$$\begin{aligned} \tan(x_M) \le \tan(2\epsilon) &\iff \frac{3\sin(\epsilon)\cos(\epsilon)}{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2} \le \frac{2\sin(\epsilon)\cos(\epsilon)}{\cos(\epsilon)^2 - \sin^2(\epsilon)} \\ &\iff \frac{3}{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2} \le \frac{2}{\cos(\epsilon)^2 - \sin^2(\epsilon)} \\ &\iff \frac{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2}{3} \ge \frac{\cos(\epsilon)^2 - \sin^2(\epsilon)}{2} \\ &\iff \frac{2\cos(\epsilon)^2}{6} \ge \frac{\sin^2(\epsilon)}{2} \iff \tan(\epsilon)^2 \le \frac{2}{3} \end{aligned}$$

which is true for $N \ge 7$. Hence for any $N > 6$, $x_M < 3\theta_N$. Notice moreover that $f(0) = \frac{3}{2}\sin(\epsilon)^2 + \cos(\epsilon)^2 \ge 0$ hence the map is maximal at x_M. As we are only considering even multiples $2k\theta_N$, by the symmetry property, the map

$$\frac{3}{2}\sin(3\theta_N/2)\sin((2k - \frac{3}{2})\theta_N) + \cos(3\theta_N/2)^2\cos(2k\theta_N)$$

is maximal for $k = 1$ with value

$$\begin{aligned} &\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) + \cos(3\theta_N/2)^2\cos(2\theta_N) \\ \le\ &\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) + \cos(3\theta_N/2)\cos(\theta_N/2). \end{aligned}$$

For the cases $N = 4, 5, 6$ one only has to check the inequality for $k = 2$. This can be done directly.

4.6. Computing with v_1

4.6.1. Setup. Recall that the vector v_1 is the second one in the cone, so we have

$$v_1 = \begin{bmatrix} -L_2 cc_1 \\ -\frac{cc_1^2}{s_1} \\ L_1 cc_2 \\ \frac{cc_2^2}{c_2} \end{bmatrix} \text{ so } (-R)^k v_1 = \begin{bmatrix} -L_2 \cdot cc_1 \cdot cp_1 + \frac{cc_1^2}{s_1} \cdot sp_1 \\ -L_2 \cdot cc_1 \cdot sp_1 - \frac{cc_1^2}{s_1} \cdot cp_1 \\ L_1 \cdot cc_2 \cdot cp_2 - \frac{cc_2^2}{c_2} \cdot sp_2 \\ L_1 \cdot cc_2 \cdot sp_2 + \frac{cc_2^2}{c_2} \cdot cp_2 \end{bmatrix}$$

Computing now $MA(-R)^k v_1$ we find:

$$\begin{bmatrix} L_1 L_2 (cp_2 - cp_1) + L_1 cc_1 \frac{sp_1}{s_1} - L_2 cc_2 \frac{sp_2}{s_2} \\ L_2^2 sp_1 s_1 - L_1^2 sp_2 s_2 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \\ L_1 cp_2 - L_2 cp_1 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2} \\ -(L_2+1) cc_1 cp_1 + (L_1+1) cc_2 cp_2 - L_2 sp_1 s_1 + L_1 sp_2 s_2 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \end{bmatrix}$$

4.6.2. The first entry of $MA(-R)^k v_1$. Notice that it is similar to the second entry for v_0. As in §4.4.6 we split the proof into two cases. The proof is very similar but exchanges the two cases.

4.6.3. The case $N/2 \le k < N$. Notice that as $L_2 > L_1$, $1 - L_1 > 1 - L_2$ and $-cp_2 \ge -cp_1$ on this range since $cp_2 - cp_1 = 2\sin(k\theta_N)\sin(2k\theta_N) \le 0$. Multiplying these three inequalities, we get

$$-L_2(1 - L_1)cp_2 \ge -L_1(1 - L_2)cp_1$$

then

$$L_1 L_2 cp_2 - L_1 L_2 cp_1 \ge L_2 cp_2 - L_1 cp_1.$$

Hence the entry is not larger than

$$L_1\left(cc_1 \frac{sp_1}{s_1} - cp_1\right) - L_2\left(cc_2 \frac{sp_2}{s_2} - cp_2\right)$$

which is exactly the fourth coordinate of v_0, proved to be positive in §4.4.5.

4.6.4. The case $1 \leq k \leq N/2$. We compare the first entry with

$$L_1(cp_1 + cc_1\frac{sp_1}{s_1}) - L_2(cp_2 + cc_2\frac{sp_2}{s_2}).$$

which was proved to be positive in §4.4.8. Subtracting the above expression from the entry of interest, it suffices to show that

$$L_2(L_1+1)cp_2 - L_1(L_2+1)cp_1 \geq 0.$$

Using identities for L_i from Eqn. (4.2.9) and factoring out $3(L_1+L_2)$ we reduce to

$$L_2\frac{cp_2}{cc_2} - L_1\frac{cp_1}{cc_1} \geq 0.$$

The factor $\frac{cp_2}{cc_1}$ is positive in this range of k, so we reduced to showing:

$$L_2\frac{cc_1}{cc_2} - L_1\frac{cp_1}{cp_2} \geq 0$$

This was done in §4.4.7.

4.6.5. The third entry of $MA(-R)^k v_1$. This entry is equal to

$$\begin{aligned}
&L_1cp_2 - L_2cp_1 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} = \\
&\quad = (L_1+1)cp_2 - (L_2+1)cp_1 + cp_1 - cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} \\
&\quad = 3(L_1+L_2)\left(\frac{cp_2}{cc_2} - \frac{cp_1}{cc_1}\right) + \left(cp_1 + cc_1\frac{sp_1}{s_1}\right) - \left(cp_2 + cc_2\frac{sp_2}{s_2}\right)
\end{aligned}$$

Using the formulas from §4.4.8 we get

$$= 3(L_1+L_2)\frac{cp_2}{cc_1}\left(\frac{cc_1}{cc_2} - \frac{cp_1}{cp_2}\right) - \frac{\sin\left(3(k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)} + \frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)}$$

with the angle tripling formula

$$\begin{aligned}
&= 6(L_1+L_2)\frac{cp_2}{cc_1}\left(\cos(2\theta_N) - \cos(2k\theta_N)\right) \\
&\quad + 2\cdot\frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\left(\cos(\theta_N) - \cos\left((2k-1)\theta_N\right)\right).
\end{aligned}$$

Using the difference of cosines formula, and factoring out 4, we get

$$\begin{aligned}
&3(L_1+L_2)\frac{cp_2}{cc_1}\sin\left((k+1)\theta_N\right)\sin\left((k-1)\theta_N\right) \\
&+\frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\sin\left(k\theta_N\right)\sin\left((k-1)\theta_N\right).
\end{aligned}$$

Factoring out $\frac{\sin((k-1)\theta_N)}{cc_1\sin(3\theta_N/2)}$, we get:

$$3(L_1+L_2)\cdot cp_2\cdot\sin\Big((k+1)\theta_N\Big)\cdot\sin(3\theta_N/2)$$
$$+2\cdot\sin\left((k-\frac{1}{2})\theta_N\right)\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2.$$

For $1\leq k\leq N/2$ this expression is clearly positive. Let us deal with $N/2<k<N$. In this case, the first term of the sum is negative and the second positive. Using the fact that $L_1+L_2<1$ and $\sin\Big((k-\frac{1}{2})\theta_N\Big)>\sin\left((k+1)\theta_N\right)$ on this domain we have the following lower bound

$$\sin\Big((k+1)\theta_N\Big)\Big(3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2)+2\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2\Big).$$

Factoring out the sine factor and using the fact that for $N\geq 5$, $\cos(3\theta_N/2)\geq\frac{1}{2}$ we observe that

$$\begin{aligned}3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2)&+2\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2\\
\geq 3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2)&+\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)\\
&\geq 3\sin((k+\frac{3}{2})\theta_N)-2\cos(3\theta_N/2)\cdot\sin(k\theta_N)\\
&\geq 3\sin((k+\frac{3}{2})\theta_N)-\sin(k\theta_N)\\
&\geq 3\sin(k\theta_N)-\frac{3}{2}\theta_N-\sin(k\theta_N)\\
&\geq 2\sin(k\theta_N)-\frac{3}{2}\theta_N.\end{aligned}$$

This last expression is minimal for $k=N-1$. We are then reduced to showing that $\sin(\theta_N)\geq\frac{3}{4}\theta_N$. This is true for $N>4$ since $\cos(x)\geq\frac{3}{4}$ for all $x\in[0,\frac{\pi}{5}]$ and can be checked directly for $N=4$.

4.6.6. The fourth entry of $MA(-R)^k v_1$. With the identity

$$cc_i^2+s_i^2=2cc_i$$

for $i\in\{0,1\}$, we can rewrite the fourth coordinate as

$$\begin{aligned}-(L_2+1)cc_1cp_1+(L_1+1)cc_2cp_2-(L_2+1)sp_1s_1+(L_1+1)sp_2s_2+2\left(cc_1\frac{sp_1}{s_1}-cc_2\frac{sp_2}{s_2}\right)\\
=3(L_1+L_2)\left(-cp_1+cp_2-\frac{s_1}{cc_1}sp_1+\frac{s_2}{cc_2}sp_2\right)+2\left(cc_1\frac{sp_1}{s_1}-cc_2\frac{sp_2}{s_2}\right).\end{aligned}$$

4.6.7. The case $1\leq k\leq N/2$. Notice that the second term of the sum is exactly the fourth coordinate for v_3 proved to be positive in 4.5.4.

The first term of the sum can be rewritten, using Proposition 4.2.20, as

$$4\cdot 3(L_1+L_2)\frac{\cos\left((k+\frac{1}{2})\theta_N\right)}{\cos(3\theta_N/2)}\sin\left((k+1)\theta_N\right)\sin(k\theta_N)$$

which is positive for $1\leq k<N/2$.

4.6.8. The case $N/2\leq k<N$. In this case

$$3(L_1+L_2)\left(-cp_1+cp_2-\frac{s_1}{cc_1}sp_1+\frac{s_2}{cc_2}sp_2\right)\geq 3\left(-cp_1+cp_2-\frac{s_1}{cc_1}sp_1+\frac{s_2}{cc_2}sp_2\right)$$

Using the previous computation and §4.5.4, we are reduced to showing positivity of

$$\begin{aligned}4\cdot 3\frac{\cos\left((k+\frac{1}{2})\theta_N\right)}{\cos(3\theta_N/2)}\sin\left((k+1)\theta_N\right)\sin(k\theta_N)\\ +2\cdot 2\sin(k\theta_N)\left(\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cos(2k\theta_N)\right).\end{aligned}$$

Factoring $\frac{4\sin(k\theta_N)}{\sin(3\theta_N/2)\cos(3\theta_N/2)}$ out yields:

$$\begin{aligned}3\sin(3\theta_N/2)\cos\left((k+\frac{1}{2})\theta_N\right)\sin((k+1)\theta_N)\\ +\cos(3\theta_N/2)\left(\cos(\theta_N/2)-\cos(3\theta_N/2)\cos(2k\theta_N)\right).\end{aligned}$$

Using the relation

$$2\cos((k+\frac{1}{2})\theta_N)\sin((k+1)\theta_N)=\sin\left((2k+\frac{3}{2})\theta_N\right)+\sin\left(\theta_N/2\right),$$

we can rewrite

$$\begin{aligned}\frac{3}{2}\sin(3\theta_N/2)\sin\left((2k+\frac{3}{2})\theta_N\right)-\cos(3\theta_N/2)^2\cos\left(2k\theta_N\right)\\ +\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2)+\cos(3\theta_N/2)\cos(\theta_N/2)\end{aligned}$$

which positivity is equivalent to

$$\begin{aligned}\frac{3}{2}\sin(3\theta_N/2)\cos\left((2k+\frac{3}{2})\theta_N\right)-\cos(3\theta_N/2)^2\cos\left(2k\theta_N\right)\\ \geq -\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2)-\cos(3\theta_N/2)\cos(\theta_N/2).\end{aligned}$$

We now need to understand the minimal value of the function

$$\begin{aligned}\frac{3}{2}\sin(3\theta_N/2)\sin\left((2k+\frac{3}{2})\theta_N\right)-\cos(3\theta_N/2)^2\cos\left(2k\theta_N\right)=\\ a\sin(x-\epsilon)+b\cos(x)=f(x)\end{aligned}$$

where $a = \frac{3}{2}\sin(3\theta_N/2)$, $b = -\cos(3\theta_N/2)^2$ and $\epsilon = 3\theta_N/2$. Then according to Proposition 4.5.8, if x_M is a point on which the function is extremal, then

$$\begin{aligned}\tan(x_M) &= \frac{a\cos(\epsilon)}{b - a\sin(\epsilon)} \\ &= \frac{3\sin(\epsilon)\cos(\epsilon)}{-2\cos(\epsilon)^2 - 3\sin(\epsilon)^2}.\end{aligned}$$

Hence

$$\begin{aligned}\tan(-x_M) \leq \tan(2\epsilon) &\iff \frac{3\sin(\epsilon)\cos(\epsilon)}{2\cos(\epsilon)^2 + 3\sin(\epsilon)^2} \leq \frac{2\sin(\epsilon)\cos(\epsilon)}{\cos(\epsilon)^2 - \sin(\epsilon)^2} \\ &\iff \frac{3}{2\cos(\epsilon)^2 + 3\sin(\epsilon)^2} \leq \frac{2}{\cos(\epsilon)^2 - \sin(\epsilon)^2} \\ &\iff \frac{2\cos(\epsilon)^2 + 3\sin(\epsilon)^2}{3} \geq \frac{\cos(\epsilon)^2 - \sin(\epsilon)^2}{2} \\ &\iff 4\cos(\epsilon)^2 + 6\sin(\epsilon)^2 \geq 3\cos(\epsilon)^2 - 3\sin(\epsilon)^2\end{aligned}$$

which is true. Hence $x_M \in (\pi - 2\epsilon, \pi)$ and moreover as $f(0) = -\frac{3}{2}\sin(\epsilon)^2 - \cos(\epsilon)^2 < 0$, x_M is the maximum of f and its minimum is at $x_m = x_M + \pi \in (2\pi - 2\epsilon, 2\pi)$. Thus the above expression takes its minimum value for $k = N-1$ and is bounded from below by

$$\begin{aligned}\frac{3}{2}\sin(3\theta_N/2)\cos\left((2(N-1) + \frac{3}{2})\theta_N\right) &- \cos(3\theta_N/2)^2\cos\left(2(N-1)\theta_N\right) \\ \frac{3}{2}\sin(3\theta_N/2)\cos\left(\theta_N/2\right) &- \cos(3\theta_N/2)^2\cos\left(\theta_N\right) \\ \geq -\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) &- \cos(3\theta_N/2)\cos(\theta_N/2).\end{aligned}$$

The last inequality follows from the fact that $\cos(3\theta_N/2)^2\cos(\theta_N) \leq \cos(3\theta_N/2)\cos(\theta_N/2)$.

4.6.9. The second entry of $MA(-R)^k v_1$.

$$\begin{aligned}&L_2^2 sp_1 s_1 - L_1^2 sp_2 s_2 + cc_1^2\frac{sp_1}{s_1} - cc_2^2\frac{sp_2}{s_2} \\ &= (L_2^2 s_1 + \frac{cc_1^2}{s_1})sp_1 - (L_1^2 s_2 + \frac{cc_2^2}{s_2})sp_2.\end{aligned}$$

Factoring out $\sin(k\theta_N)$ we get

$$-(L_2^2 s_1 + \frac{cc_1^2}{s_1})\left(2\cos(2k\theta_N) + 1\right) + (L_1^2 s_2 + \frac{cc_2^2}{s_2}).$$

As cosine is decreasing on $[0, \pi]$, and increasing on $[\pi, 2\pi]$, we only have to check that

$$-(L_2^2 s_1 + \frac{cc_1^2}{s_1})\,(2\cos(2\theta_N) + 1) + (L_1^2 s_2 + \frac{cc_2^2}{s_2}) \geq 0$$

and

$$-(L_2^2 s_1 + \frac{cc_1^2}{s_1})\,(2\cos(2(N-1)\theta_N) + 1) + (L_1^2 s_2 + \frac{cc_2^2}{s_2}) \geq 0.$$

These two inequalities are equivalent to

$$\begin{aligned} \frac{L_1^2 s_2 + \frac{cc_2^2}{s_2}}{L_2^2 s_1 + \frac{cc_1^2}{s_1}} &\geq 2\cos(2\theta_N) + 1 \\ \iff \frac{s_1}{s_2} \cdot \frac{L_1^2 s_2^2 + cc_2^2}{L_2^2 s_1^2 + cc_1^2} &\geq 2\cos(2\theta_N) + 1 \\ \iff (2\cos(3\theta_N/2) + 1)\,\frac{L_1^2 s_2^2 + cc_2^2}{L_2^2 s_1^2 + cc_1^2} &\geq 2\cos(2\theta_N) + 1. \end{aligned}$$

Again as cosine is decreasing on $[0, \pi]$, we only need to show

$$\begin{aligned} L_1^2 s_2^2 + cc_2^2 &\geq L_2^2 s_1^2 + cc_1^2 \\ \iff (cc_1 + cc_2)(cc_2 - cc_1) &\geq (L_2 s_1 + L_1 s_2)(L_2 s_2 - L_1 s_1) \end{aligned}$$

Notice that $(L_2 s_1 + L_1 s_2)(L_2 s_2 - L_1 s_1) \leq s_2$ and $(cc_1 + cc_2)(cc_2 - cc_1) = 2(cc_1 + cc_2)\sin(\theta_N)\sin(2\theta_N)$. Hence the inequality follows from the fact that $2(cc_1 + cc_2) \geq 1$.

4.7. Computing with Bv_1

4.7.1. Setup. We finally have to handle the vector Bv_1, namely

$$Bv_1 = \begin{bmatrix} L_2 \cdot cc_1 \\ -\frac{cc_1^2}{s_1} \\ -L_1 \cdot cc_2 \\ \frac{cc_2^2}{s_2} \end{bmatrix} \text{ so } (-R)^k Bv_1 = \begin{bmatrix} L_2 cc_1 \cdot cp_1 + \frac{cc_1^2}{s_1} sp_1 \\ L_2 cc_1 \cdot sp_1 - \frac{cc_1^2}{s_1} \cdot cp_1 \\ -L_1 cc_2 \cdot cp_2 - \frac{cc_2^2}{s_2} \cdot sp_2 \\ -L_1 cc_2 \cdot sp_2 + \frac{cc_2^2}{s_2} \cdot cp_2 \end{bmatrix}$$

Computing now $MA(-R)^k Bv_1$ we find:

$$\begin{bmatrix} L_1 L_2 (cp_1 - cp_2) + L_1 cc_1 \frac{sp_1}{s_1} - L_2 cc_2 \frac{sp_2}{s_2} \\ (L_2 + 1) cc_1 cp_1 - (L_1 + 1) cc_2 cp_2 + L_1^2 sp_2 s_2 - L_2^2 sp_1 s_1 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \\ L_2 cp_1 - L_1 cp_2 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2} \\ (L_2 - 1) cc_1 cp_1 - (L_1 - 1) cc_2 cp_2 + L_2 sp_1 s_1 - L_1 sp_2 s_2 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \end{bmatrix}$$

4.7.2. The first entry of $MA(-R)^k Bv_1$. Notice that it corresponds to the second entry for v_0 proved to be positive in §4.4.6.

4.7.3. The third entry of $M_{test}(-R)^k Bv_1$. This entry is very similar to the one in §4.6.5 and we follow the same scheme of proof.

$$\begin{aligned}
&L_2 cp_1 - L_1 cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}\\
&\quad = (L_2+1)cp_1 - (L_1+1)cp_2 + cp_2 - cp_1 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}\\
&\quad = 3(L_1+L_2)\left(\frac{cp_1}{cc_1} - \frac{cp_2}{cc_2}\right) - \left(cp_1 - cc_1\frac{sp_1}{s_1}\right) + \left(cp_2 - cc_2\frac{sp_2}{s_2}\right)
\end{aligned}$$

Using formulas from §4.4.5 we get

$$= 3(L_1+L_2)\frac{cp_2}{cc_1}\left(\frac{cp_1}{cp_2} - \frac{cc_1}{cc_2}\right) - \frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)} + \frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)}$$

with the angle tripling formula

$$\begin{aligned}
&= -6(L_1+L_2)\frac{cp_2}{cc_1}\left(\cos(2\theta_N) - \cos(2k\theta_N)\right)\\
&\quad + 2\cdot\frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\left(\cos(\theta_N) - \cos\left((2k+1)\theta_N\right)\right).
\end{aligned}$$

Using the difference of cosines formula, and factoring out 4, we get

$$\begin{aligned}
&-3(L_1+L_2)\frac{cp_2}{cc_1}\sin\left((k+1)\theta_N\right)\sin\left((k-1)\theta_N\right)\\
&\quad + \frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\sin\left(k\theta_N\right)\sin\left((k-1)\theta_N\right).
\end{aligned}$$

Factoring be $\frac{\sin((k-1)\theta_N)}{cc_1\sin(3\theta_N/2)}$,

$$\begin{aligned}
&-3(L_1+L_2)\cdot cp_2\cdot\sin\left((k+1)\theta_N\right)\cdot\sin(3\theta_N/2)\\
&\quad +2\cdot\sin\left((k+\frac{1}{2})\theta_N\right)\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2.
\end{aligned}$$

For $N/2 \le k < N$ this expression is clearly positive. Let us deal with $1 \le k < N/2$. In this case, the first term of the sum is negative and the second positive. Using the fact that $L_1 + L_2 < 1$ and $\sin\left((k+1)\theta_N\right) < \sin\left((k+\frac{1}{2})\theta_N\right)$ on this domain we have the following lower bound

$$\sin\left((k+\frac{1}{2})\theta_N\right)\left(-3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2) + 2\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2\right).$$

Factoring out the first and using the fact that for $N \geq 5$, $\cos(3\theta_N/2) \geq \frac{1}{2}$ we observe that

$$\begin{aligned}
-3 \cdot \cos(k\theta_N) \cdot \sin(3\theta_N/2) + \sin(k\theta_N) \cdot \cos(3\theta_N/2) \\
\geq 3\sin((k - \frac{3}{2})\theta_N) - 2\cos(3\theta_N/2) \cdot \sin(k\theta_N) \\
\geq 3\sin((k - \frac{3}{2})\theta_N) - \sin(k\theta_N) \\
\geq 3\sin(k\theta_N) - \frac{3}{2}\theta_N - \sin(k\theta_N) \\
\geq 2\sin(k\theta_N) - \frac{3}{2}\theta_N.
\end{aligned}$$

This last expression is minimal for $k = 1$. We are then reduced to showing that $\sin(\theta_N) \geq \frac{3}{4}\theta_N$. This was proved to be true for all $N \geq 4$ in §4.6.5.

4.7.4. The second entry of $MA(-R)^kBv_1$. Let us give an alternative expression for the second entry.

$$\begin{aligned}
&(L_2+1)cc_1cp_1 - (L_1+1)cc_2cp_2 + L_1^2 sp_2 s_2 - L_2^2 sp_1 s_1 + cc_1^2\frac{sp_1}{s_1} - cc_2^2\frac{sp_2}{s_2} \\
&= 3(L_1+L_2)(cp_1 - cp_2) + (L_1^2+1)sp_2s_2 - (L_2^2+1)sp_1s_1 + 2cc_1\frac{sp_1}{s_1} - 2cc_2\frac{sp_2}{s_2}
\end{aligned}$$

Factoring $\sin(k\theta_N)$ out, we get

$$\begin{aligned}
-6(L_1+L_2)\sin(2k\theta_N) + \left(2\frac{cc_2}{s_2} - (L_1^2+1)s_2\right) - \left(2\frac{cc_1}{s_1} - (L_2^2+1)s_1\right)(2\cos(2k\theta_N)+1) \\
= \left(2\frac{cc_2}{s_2} - (L_1^2+1)s_2\right) - \left(2\frac{cc_1}{s_1} - (L_2^2+1)s_1\right) \\
-2\left(\left(2\frac{cc_1}{s_1} - (L_2^2+1)s_1\right)\cos(2k\theta_N) + 3(L_1+L_2)\sin(2k\theta_N)\right)
\end{aligned}$$

Let us determine the maximal value of

$$\left(2\frac{cc_1}{s_1} - (L_2^2+1)s_1\right)\cos(2k\theta_N) + 3(L_1+L_2)\sin(2k\theta_N) = f(2k\theta_N).$$

There exists a unique $x_M \in [0, \pi)$ such that $f(x_M)$ is maximal since $L_1 + L_2 > 0$ and it satisfies

$$\tan(x_M) = \frac{3(L_1+L_2)s_1}{2cc_1 - (L_2^2+1)s_1} \leq \frac{3s_1}{2 - 2s_1^2} = \frac{3}{2c_1}\tan(\theta_N).$$

For $N \geq 5$, $\cos(3\pi/2N) \geq \frac{1}{2}$, hence

$$\tan(x_M) \leq 3\tan(\theta_N).$$

By convexity of the tangent function on $[0, \frac{\pi}{2})$ this is less than $\tan(2\theta_N)$. This implies that the maximum of the entry is reached at $k = 1$. Hence the minimal value of the entry on this domain is also reached at $k = 1$ which evaluated on the first expression gives

$$\begin{aligned}
&3(L_1 + L_2)(c_1 - c_2) - L_1^2 s_2^2 + L_2^2 s_1^2 - cc_1^2 + cc_2^2 \\
&= 3(L_1 + L_2)(c_1 - c_2) + (cc_1 + cc_2)(c_2 - c_1) + L_2^2 s_1^2 - L_1^2 s_2^2 \\
&\geq (cc_1 + cc_2 - 3)(c_2 - c_1).
\end{aligned}$$

This last expression is positive, since $\cos(3\theta_N)+\cos(\theta_N) = 2\cos(\theta_N)(\cos(2\theta_N)+2)$ which is increasing with N and equal to $\frac{4}{\sqrt{2}} > 1$ for $N = 4$.

4.7.5. The fourth entry of $MA(-R)^k Bv_1$. We rewrite it as:

$$\begin{aligned}
&(L_2 - 1)cc_1cp_1 - (L_1 - 1)cc_2cp_2 + L_2 sp_1 s_1 - L_1 sp_2 s_2 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \\
&= (1 - L_1)cc_2cp_2 - (1 - L_2)cc_1cp_1 + (1 - L_1)sp_2 s_2 - (1 - L_2)sp_1 s_1 \\
&\qquad + 2\left(cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2}\right) \\
&= 3(L_1 + L_2)\left(cp_1 - cp_2 + sp_1 \frac{s_1}{cc_1} - sp_2 \frac{s_2}{cc_2}\right) \\
&\qquad -2(cc_1cp_1 - cc_2cp_2 + sp_1 s_1 - sp_2 s_2) + 2\left(cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2}\right)
\end{aligned}$$

4.7.6. The case $1 \leq k \leq N/2$. We compute the difference with the second entry

$$\begin{aligned}
&-2cc_1cp_1 + 2cc_2cp_2 + L_2(L_2 + 1)sp_1 s_1 - L_1(L_1 + 1)sp_2 s_2 \\
&= 2(cc_2 - cc_1)cp_2 + 2cc_1(cp_2 - cp_1) + L_2(L_2 + 1)sp_1 s_1 - L_1(L_1 + 1)sp_2 s_2.
\end{aligned}$$

Hence we need to check the following.

$$\begin{aligned}
&2(cc_2 - cc_1)cp_2 + 2cc_1(cp_2 - cp_1) + L_2(L_2 + 1)sp_1 s_1 - L_1(L_1 + 1)sp_2 s_2 \geq 0 \\
&\iff 2(cc_2 - cc_1)\cos(k\theta_N) \geq \\
&\sin(k\theta_N)\left(-4cc_1 \sin(2k\theta_N) + L_2(L_2 + 1)s_1(2\cos(2k\theta_N) + 1) - L_1(L_1 + 1)s_2\right).
\end{aligned}$$

The left-hand side term is non-negative on the domain and we show in the following that the right-and side term is non-positive. Factoring $\sin(k\theta_N)$ out,

$$\begin{aligned}
&-4cc_1 \sin(2k\theta_N) + L_2(L_2 + 1)s_1(2\cos(2k\theta_N) + 1) - L_1(L_1 + 1)s_2 \\
&= -4cc_1 \sin(2k\theta_N) + 2L_2(L_2 + 1)s_1 \cos(2k\theta_N) + L_2(L_2 + 1)s_1 - L_1(L_1 + 1)s_2 \\
&\geq -4cc_1 \sin(2k\theta_N) + 2L_2(L_2 + 1)s_1 \cos(2k\theta_N).
\end{aligned}$$

Notice that this last expression is positive for $k = 0$ and for $k = \frac{3}{2}$, it is equal to

$$-4cc_1 s_1 + 2L_2(L_2+1)s_1(cc_1 - 1) \leq -4cc_1 s_1 + 2s_1 cc_1 < 0.$$

As in Proposition 4.5.8 this expression can be expressed as a shifted cosine $A\cos(2k\theta_N - D)$ for which $A \leq 0$ and $0 < D < 3\theta_N$. Hence it is negative for $1 < k \leq N/2$. We conclude by noticing that the initial expression is clearly positive for $k = 1$, since $cc_1 < cc_2$, $\cos(3\theta_N) < \cos(\theta_N)$, $L_1 < L_2$ and $s_2 < s_1$.

4.7.7. The case $N/2 < k < N$. First notice that, by Proposition 4.2.20,

$$cp_1 - cp_2 + sp_1\frac{s_1}{cc_1} - sp_2\frac{s_2}{cc_2} = -4\cdot\frac{\cos((k+\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}\cdot\sin(k\theta_N)\cdot\sin((k+1)\theta_N)$$

which is non negative on the domain. Thus we are reduced to showing positivity of

$$\frac{cc_1}{s_1}sp_1 - cc_1 cp_1 - sp_1 s_1 - \frac{cc_2}{s_2}sp_2 + cc_2 cp_2 + sp_2 s_2.$$

Now, notice that

$$\begin{aligned}
\frac{cc_1}{s_1}sp_1 - cc_1 cp_1 - sp_1 s_1 &= \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\left(-\sin(3k\theta_N)\right) - \left(1+\cos(3\theta_N)\right)cp_1 - sp_1 s_1 \\
&= \frac{1}{\sin(3\theta_N/2)}\Big(\cos(3\theta_N/2)(-\sin(3k\theta_N)) - \cos(3k\theta_N)\sin(3\theta_N/2)\Big) \\
&\qquad - \cos(3\theta_N)\cos(3k\theta_N) + \sin(3k\theta_N)\sin(3\theta_N) \\
&= \frac{-\sin(3(k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} - \cos(3(k+1)\theta_N).
\end{aligned}$$

Similarly, we have

$$\frac{cc_2}{s_2}sp_2 - cc_2 cp_2 - sp_2 s_2 = \frac{-\sin((k+\frac{1}{2})\theta_N)}{\sin(\theta_N/2)} - \cos((k+1)\theta_N).$$

Subtracting the two terms, we get

$$\begin{aligned}
&\frac{-\sin(3(k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2} + \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(\theta_N/2)} - \cos(3(k+1)\theta_N) + \cos((k+1)\theta_N) \\
= &\frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\left(2\cos(\theta_N) - 2\cos((2k+1)\theta_N)\right) - \cos(3(k+1)\theta_N) + \cos((k+1)\theta_N)
\end{aligned}$$

Using the formula for sum of cosines,

$$
\begin{aligned}
&= 4 \cdot \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \sin(k\theta_N)\sin((k+1)\theta_N) + 2\cdot \sin((k+1)\theta_N)\sin(2(k+1)\theta_N) \\
&= 4 \cdot \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \sin(k\theta_N)\sin((k+1)\theta_N) + 4\cdot \sin((k+1)\theta_N)^2\cos((k+1)\theta_N) \\
&= 4\cdot \sin((k+1)\theta_N)\left(\frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\sin(k\theta_N) + \sin((k+1)\theta_N)\cos((k+1)\theta_N)\right).
\end{aligned}
$$

The sine function is decreasing on this domain, hence we bound from below by

$$
4\sin((k+1)\theta_N)^2\left(\frac{\sin(k\theta_N)}{\sin(3\theta_N/2)} + \cos((k+1)\theta_N)\right).
$$

We are thus reduced to showing positivity for

$$
\sin(k\theta_N) + \sin(3\theta_N/2)\cos((k+1)\theta_N)
$$

in the range $k > \frac{N}{2}$ and $k+1 < N$. As this last expression is decreasing on the domain, we only have to check it for $k = N-2$, *i.e.*

$$
\sin(2\theta_N) - \sin(3\theta_N/2)\cos(\theta_N) \geq 0.
$$

This is implied by the fact that $\sin(2\theta_N) \geq \sin(3\theta_N/2)$.

5. Symplectic Geometry and Causality

Outline of section. Introduced by Drumm in [Dru92], crooked surfaces are used in Lorenzian geometry to produce fundamental domains for group actions, see e.g. [DGK16]. The basic definitions and constructions are introduced in §5.1. In §5.2 we will reinterpret some of the disjointness criteria for crooked surfaces from [BCFG21] using cones. We extend their analysis to situations when crooked surfaces can touch, a geometric situation that occurs in our case. Finally, in §5.3, we will construct a domain of discontinuity for the action of Γ_N on $\mathrm{LGr}(V)$. The construction will be in two stages: first an open set Ω° built directly from the definition of crooked surfaces, then a larger domain Ω where we have added some sets where the crooked surfaces "touch", but on which the action is nonetheless properly discontinuous.

General conventions. To lighten the notation, when it is clear from the context a nonzero element in a vector space and the induced line in the projectivization will carry the same notation.

5.1. Crooked surfaces

5.1.1. Symplectic conventions. Let V be a real 4-dimensional symplectic vector space. Fix a basis e_1, f_1, e_2, f_2 such that the symplectic pairing denoted by I satisfies

$$I(e_1, f_1) = I(e_2, f_2) = 1$$

Fix also an anti-symplectic involution A given by the formula:

$$Ae_i = -e_i \quad Af_i = -f_i$$

Let us note for convenience of reference that our basis is related to the one used in [BCFG21, §5] by:

$$\begin{aligned} e_1 &= u_+ & f_1 &= -u_- \\ e_2 &= v_+ & f_2 &= v_- \end{aligned} \tag{5.1.2}$$

Note in particular the minus sign in front of u_-, which we hope minimizes the number of further negative signs later.

Given two vectors $v, v' \in V$, not proportional, we will denote by $L_{vv'}$ their 2-dimensional span or its projectivization. Typically we will consider the case when this is a Lagrangian.

5.1.3. The cone. In analogy with our constructions in previous sections, we will consider the cone

$$\mathcal{C} := \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}f_1 + \mathbb{R}_{\geq 0}f_2$$

We regard this as a projective cone $\mathcal{C} \subset \mathbb{P}(V_{\mathbb{R}})$, and later will denote by $\mathring{\mathcal{C}}$ the interior of the cone, also in projective space. The projective cone is a tetrahedron with four of the edges contained in the Lagrangian planes (projective lines):

$$\begin{aligned} L_{e_1e_2} &:= \operatorname{span}(e_1, e_2) & L_{e_1f_2} &:= \operatorname{span}(e_1, f_2) \\ L_{f_1f_2} &:= \operatorname{span}(f_1, f_2) & L_{e_2f_1} &:= \operatorname{span}(e_2, f_1) \end{aligned}$$

The remaining two edges are contained in the projectivization of subspaces orthogonal for the symplectic form, and on which the symplectic form is non-degenerate:

$$S_1 = \operatorname{span}(e_1, f_1) \quad S_2 = \operatorname{span}(e_2, f_2)$$

5.1.4. Indefinite inner product conventions. Let now $W := \Lambda^2_0 V$ denote the subspace of the second exterior power which wedges to zero against $e_1 \wedge f_1 + e_2 \wedge f_2$ (or equivalently is in the kernel of the symplectic form). Then W is equipped with a nondegenerate quadratic form of signature $(2, 3)$ given by taking the wedge product of elements and using the trivialization of $\Lambda^4 V$ by the volume form induced from the symplectic pairing. Given these sign conventions, we will say that a subspace is time-like if it is positive definite, and space-like if it is negative definite.

We will use the explicit basis of W given by

$$e_1 \wedge e_2 \quad e_1 \wedge f_2 \quad e_1 \wedge f_1 - e_2 \wedge f_2 \quad f_1 \wedge e_2 \quad f_1 \wedge f_2$$

In this basis the induced involution (which actually preserves the inner product) has eigenvalue $+1$ on $e_1 \wedge e_2$ and $f_1 \wedge f_2$, and eigenvalue -1 on the remaining three basis vectors.

5.1.5. The Lagrangian Grassmannian. Recall next that the Lagrangian Grassmannian $\mathrm{LGr}(V)$ is equal to the quadric of null vectors in $\mathbb{P}(W)$:

$$\mathrm{LGr}(V) = \{[w] \in \mathbb{P}(W) \colon w^2 = 0\} \subset \mathbb{P}(W).$$

It is equipped with a conformal class of Lorenzian metrics of signature $(1,2)$ and is frequently also called an Einstein universe and denoted $\mathrm{Ein}^{1,2}$.

5.1.6. Photons. Associated to a nonzero vector $v \in V$ (rather, the corresponding point in $\mathbb{P}(V)$) there is a "photon" of Lagrangians:

$$\phi(v) := \{L \in \mathrm{LGr}(V) \colon v \in L\} \subset \mathrm{LGr}(V)$$

The photon can also be identified as

$$\phi(v) = \mathbb{P}\left(v \wedge v^\perp\right) \cong \mathbb{P}\left(v^\perp / v\right)$$

where $v^\perp \subset V$ denotes the symplectic-orthogonal to v.

5.1.7. The wings of the crooked surface. We can now define the crooked surface. It consists of two "wings" and a "stem" (the stem also decomposes into two pieces, see §5.1.9 below).

Consider the "interval of lines"

$$[e_1, e_2] := \{se_1 + te_2 \colon s, t \geq 0, s + t = 1\} \subset \mathbb{P}(V) \tag{5.1.8}$$

which is one of the boundary edges of the cone $\mathcal{C}$. Then the e-wing is defined as:

$$\mathcal{W}_e := \phi([e_1, e_2]) = \bigcup_{l \in [e_1, e_2]} \phi(l).$$

Analogously define the f-wing:

$$\mathcal{W}_f := \phi([f_1, f_2]).$$

5.1.9. The stem of the crooked surface. Consider the "Einstein torus" consisting of Lagrangians spanned by one vector in each of S_1, S_2:

$$\mathrm{Ein}^{1,1}(S_1, S_2) = \{L = w_1 \wedge w_2 \colon w_i \in S_i\} \tag{5.1.10}$$

Then the stem is defined as:

$$\mathcal{S} := \{L \in \mathrm{Ein}^{1,1}(S_1, S_2)\colon |\mathrm{Maslov}(L_{e_1f_2}, L, L_{e_2f_1})| = 2\}$$

Note that we can further decompose the stem as $\mathcal{S} = \mathcal{S}^+ \coprod \mathcal{S}^-$ according to the sign of the Maslov index (see §5.1.12 below for the definition of the Maslov index).

Set now the crooked surface to be

$$\mathcal{WS} := \mathcal{W}_e \coprod \mathcal{S} \coprod \mathcal{W}_f \tag{5.1.11}$$

Observe that according to the definitions, the wings are relatively closed subsets, while the stem is a relatively open set in $\mathcal{WS}$.

5.1.12. Making the stem explicit. Recall that the Maslov index of a Lagrangian is defined as the index of the quadratic form obtained from the symplectic form, using the direct sum decomposition provided by two transverse Lagrangians. In the case at hand $V = L_{e_1f_2} \oplus L_{e_2f_1}$ and the quadratic form, denoted Q_{12}, comes out to be

$$\begin{aligned} Q_{12}(\alpha_1e_1 + \alpha_2e_2 + \beta_1f_1 + \beta_2f_2) &:= I(\alpha_1e_1 + \beta_2f_2, \alpha_2e_2 + \beta_1f_1) \\ &= \alpha_1 \cdot \beta_1 - \alpha_2 \cdot \beta_2 \end{aligned}$$

If the Lagrangian L is spanned by $w_i \in S_i$ with coordinates

$$w_1 = \alpha_1e_1 + \beta_1f_1 \quad w_2 = \alpha_2e_2 + \beta_2f_2$$

then we observe that w_1 and w_2 are orthogonal with respect to Q_{12} and $Q_{12}(w_i) = (-1)^{i+1}\alpha_i \cdot \beta_i$. So for the Lagrangian to belong to the stem both products have to be of opposite sign. We thus have:

$L = w_1 \wedge w_2$ belongs to:	$\mathcal{S}^+$	$\mathcal{S}^-$
$w_1 = \alpha_1e_1 + \beta_1f_1$	$\alpha_1 \cdot \beta_1 > 0$	$\alpha_1 \cdot \beta_1 < 0$
$w_2 = \alpha_2e_2 + \beta_2f_2$	$\alpha_2 \cdot \beta_2 < 0$	$\alpha_2 \cdot \beta_2 > 0$

5.1.13. Action of reflection. The anti-symplectic involution A preserves the crooked surface as a set. Furthermore it exchanges the two components of the stem: $A\mathcal{S}^\pm = \mathcal{S}^\mp$ and fixes as a set each photon on the wings. On individual photons on the wings, it fixes two points and exchanges the two complementary regions. Explicitly, on the photon $\phi(s \cdot e_1 + t \cdot e_2)$ belonging to the e-wing, the fixed points are the two Lagrangians $e_1 \wedge e_2$ and $(se_1 + te_2) \wedge (-tf_1 + sf_2)$. The formula for photons on the f-wing is analogous.

5.2. Disjointness of crooked surfaces

In this section, we proceed to study the geometric configurations that crooked surfaces, photons, and Lagrangians, can be in. First, some of the results from [BCFG21] can be reinterpreted using the cones that we introduced earlier. This gives transparent geometric conditions for when photons, or crooked surfaces, are disjoint. We then further refine our analysis to situations when crooked surfaces can "touch", an inevitable situation when facing groups with unipotent elements.

5.2.1. Proposition (Disjointness of photon from crooked surface)**.** *Consider a vector $v \in V$ with coordinates*

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \beta_1 f_1 + \beta_2 f_2.$$

The following are equivalent:

(i) *The photon $\phi(v) \in \mathrm{LGr}(V)$ is disjoint from the crooked surface $\mathcal{WS}$.*

(ii) *The following inequalities hold:*

$$\alpha_1 \cdot \alpha_2 > 0 \quad \textit{and} \quad \beta_1 \cdot \beta_2 > 0.$$

(iii) *The (projectivized) vector v is either in the interior of the cone $\mathring{\mathcal{C}}$ or in the interior of the reflected cone $A\mathring{\mathcal{C}}$.*

Proof. The equivalence of (i) and (ii) is simply a restatement of [BCFG21, Lemma 9]. The geometric interpretation of (ii) with cones in (iii) follows directly. Indeed the cone $\mathcal{C}$ corresponds to vectors with all coordinates of the same sign, while $A\mathcal{C}$ to those vectors where the pair $(\alpha_1 : \alpha_2)$ has the same sign, and so does $(\beta_1 : \beta_2)$, but the signs of the two pairs are opposite. □

5.2.2. Position of a Lagrangian. Let us also list the possibilities for the position of a Lagrangian relative to the crooked surface, when viewing the picture in $\mathbb{P}(V)$. Regard the Lagrangian as projectivized in $\mathbb{P}(V)$, thus yielding a line. If the Lagrangian intersects the interior of either $\mathcal{C}$ or $A\mathcal{C}$ then it is clearly in the corresponding component in $\mathrm{LGr}(V)$, since it lies on a photon entirely contained in such a component. But the Lagrangian could intersect also just the boundary, say the boundary of $\mathcal{C}$ for simplicity. If it intersects a vertex, or more generally one of the edges $[e_1, e_2]$ or $[f_1, f_2]$ then it clearly lies on the respective wing. If it intersects one of the edges $[e_1, f_1]$ or $[e_2, f_2]$, the assumption that it doesn't go through the interior of $\mathcal{C}$ implies that L must belong to one of pieces of the stem $\mathcal{S}^{\pm}$. Finally, suppose L intersects the edge $[e_1, f_2]$ (for $[e_2, f_1]$ the analysis is similar), say in $v = \alpha_1 e_1 + \beta_2 f_2$ with $\alpha_1, \beta_2 > 0$. Then its orthogonal complement is spanned by $e_1, f_2, \beta_2 f_1 +$

$\alpha_1 f_1$, and unless L is the span of e_1, f_2, it is immediate that some linear combination of vectors in L lies in the interior of $\mathcal{C}$, placing L in the interior of the respective component.

Next, we have the following criterion:

5.2.3. Proposition (Disjointness of crooked surfaces)**.** *Let $\mathcal{WS}, \mathcal{WS}$ be two crooked surfaces, with corresponding vectors e_i, f_i, e_i', f_i', cones $\mathcal{C}, \mathcal{C}'$ and anti-symplectic involutions A, A'. The following are equivalent:*

(i) *The crooked surfaces $\mathcal{WS}$ and $\mathcal{WS}'$ are disjoint.*
(ii) *The photons $\phi(e_1), \phi(e_2), \phi(f_1), \phi(f_2)$ are disjoint from $\mathcal{WS}'$ and also the photons $\phi(e_1'), \phi(e_2'), \phi(f_1'), \phi(f_2')$ are disjoint from $\mathcal{WS}$.*
(iii) *The following vectors are contained in the interiors of the cones:*

$$e_1, e_2, f_1, f_2 \in \mathring{\mathcal{C}}' \cup A' \cdot \mathring{\mathcal{C}}' \text{ and } e_1', e_2', f_1', f_2' \in \mathring{\mathcal{C}} \cup A \cdot \mathring{\mathcal{C}}.$$

Proof. Again, the equivalence of (i) and (ii) is the content of [BCFG21, Thm. 10]. The geometric interpretation in (iii) follows from Proposition 5.2.1 applied to each photon individually. ☐

Let us also recall the basic facts on the topology of the crooked surface:

5.2.4. Proposition (Connectivity and topology of crooked surface)**.** *Given a crooked surface $\mathcal{WS}$:*

(i) *It is homeomorphic to a Klein bottle: $\mathcal{WS} \approx \mathbb{K}^2$.*
(ii) *Its complement $\mathrm{LGr}(V) \setminus \mathcal{WS}$ has two connected components. The components can be labeled according to the cones $\mathcal{C}$ and $A\mathcal{C}$, corresponding to the photons which are contained in one component or the other. The components are exchanged by the anti-symplectic involution A.*

Proof. That crooked surfaces are homeomorphic to Klein bottles is [BCD$^+$08, Thm. 8.3.1]. That a crooked surface disconnects $\mathrm{LGr}(V)$ is proved in [CFLD14, Thm 3.16], and that the anti-symplectic involution exchanges the two components follows immediately as well. ☐

In our geometric applications, a "touching" of crooked surfaces occurs, because the cones can intersect along edges or faces. This situation is handled in Proposition 5.2.6 below, and we need some preliminaries on Einstein tori.

5.2.5. Einstein tori. Recall that associated to a symplectic-orthogonal splitting $V = S_1 \oplus S_2$ with symplectically non-degenerate summands, we defined an Einstein torus $\mathrm{Ein}^{1,1}(S_1, S_2)$ in Eqn. (5.1.10). As a real projective algebraic manifold it is naturally isomorphic to a product of two projective lines $\mathbb{P}(S_1) \times \mathbb{P}(S_2)$, since it is also a quadric in the projectivization of a space of signature $(2,2)$ (see also [BCD+08, §5.3]). Note that the Einstein torus embedded in $\mathbb{P}(W)$ is given as the intersection of the orthogonal complement of a negative-definite vector with the null quadric (i.e. $\mathrm{LGr}(V)$). Furthermore the torus is equipped with a natural conformal class of Lorenz metric, for which the light rays are fibers of the projection to one coordinate $\mathbb{P}^1$-factor.

Suppose given now two Einstein tori $E, E' \subset \mathrm{LGr}(V)$. Then the intersection $E \cap E'$ viewed as a subset of $E = \mathbb{P}^1 \times \mathbb{P}^1$ is a $(1,1)$-curve, i.e. cut out by a homogeneous equation of bi-degree $(1,1)$ in each of the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Three possibilities can occur for $E \cap E'$ inside E (see [BCFG21, §3]): it can be a timelike curve, it can be a spacelike curve, or it can be the union of two intersecting light rays. In the first two cases the intersection projects isomorphically to any of the $\mathbb{P}^1$-factors, in the last case each light ray projects isomorphically to a corresponding $\mathbb{P}^1$-factor.

Below is a criterion for when crooked surfaces can touch. Recall that if p, q are nonzero vectors then $[p, q]$ denotes the closed projective interval of their positive linear combinations (see Eqn. (5.1.8)) and we will denote by (p, q) the open interval where both coefficients are strictly positive.

5.2.6. Proposition (Tangency of crooked surfaces)**.** *Let $\mathcal{C}, \mathcal{C}'$ be cones determining crooked surfaces $\mathcal{WS}, \mathcal{WS}'$, with notation as in Proposition 5.2.3.*

(i) *Suppose that $[e_1, e_2] = [e_1', e_2']$ and f_1', f_2' belong to the interior $\mathcal{C}$. Then*

$$\mathcal{WS} \cap \mathcal{WS}' = \mathcal{W}_e \text{ which also equals } \mathcal{W}_e'$$

i.e. the surfaces intersect along the e-wing but nowhere else.

(ii) *Suppose that we have:*

$$f_1' = f_1 \qquad f_2' = f_2 + e_2 + \tfrac{1}{2} f_1$$
$$e_2' = e_2 + f_1 \qquad e_1' = e_1 + f_2 + \tfrac{1}{2} e_2 + \tfrac{1}{6} f_1$$

Then:

$$\mathcal{WS} \cap \mathcal{WS}' = \phi(f_1) \text{ which also equals } \phi(f_1')$$

i.e. the surfaces intersect along a photon on their f-wings but nowhere else.

The formulas in case (ii) arise when the cone $\mathcal{C}'$ is the image of $\mathcal{C}$ under a maximally unipotent symplectic matrix, which preserves the flag $f_1 \subseteq L_{f_1 e_2} \subseteq f_1^{\perp} \subseteq V$. Case (i) arises when $\mathcal{C}'$ is the image of $\mathcal{C}$ under a rank 1 symplectic unipotent matrix.

Proof. For both cases, it is immediate that the stated sets are in the intersection. We must check that no intersections occur elsewhere.

Consider case (i). First, let us see that $\mathcal{W}_f$ is disjoint from $\mathcal{WS}'$, and similarly for $\mathcal{W}'_f$ and $\mathcal{WS}$. Indeed the vectors in V spanning the photons are assumed in the interior of a cone, and so are the segments connecting them, so the disjointness of an f-wing from the (other) crooked surface follows by Proposition 5.2.1.

To see that the stems $\mathcal{S}, \mathcal{S}'$ don't intersect either, let E, E' be the Einstein tori containing them. By the discussion in §5.2.5 we see that $E \cap E' = \phi(e_1) \cup \phi(e_2)$. Indeed the two photons are clearly in the intersection (they give the joining places of the e-wings to the stem), and since the tori are distinct they account for all the intersection points. Since the stems are in the complement of these "joining" photons, their disjointness follows.

Consider now case (ii). The photons in the wing $\mathcal{W}'_e$ don't intersect $\mathcal{WS}$ in the range $[e'_1, e'_2)$ by the criterion of Proposition 5.2.1. The photon $\phi(e'_2)$ intersects $\mathcal{WS}$ at the Lagrangian $L_{e_2 f_1}$ and note except for this one point of intersection, this photon lies in the component of $\mathrm{LGr}(V) \setminus \mathcal{WS}$ corresponding to $\mathcal{C}$ (by an arbitrarily small perturbation it can be "pushed" to be entirely in the interior of that components).

The photons in the wing $\mathcal{W}'_f$ intersect $\mathcal{WS}$ at the Lagrangian $L_{f'_1 f'_2} = L_{f_1 f'_2} \in \phi(f_1) \subset \mathcal{W}_f$, but nowhere else and lie, except for this one point of intersection, in the component corresponding to $\mathcal{C}$ in $\mathrm{LGr}(V) \setminus \mathcal{WS}$ (again, a small push takes them to the interior).

Finally, the Einstein tori E, E' containing the stems intersect in two photons $E \cap E' = \phi(f_1) \cup \phi(v)$ where $v = f_2 + \frac{1}{2} e_2$ is the point of intersection between the line $f'_2 e'_2$ and the segment $[f_2, e_2]$. Let us check that the piece of the photon $\phi(v)$ that belongs to the stem $\mathcal{S}'$ is not in the stem $\mathcal{S}$. Indeed, that piece consists of Lagrangians of the form $L_{vv'}$ where $v' = \alpha'_1 e'_1 + \beta'_1 f'_1$ with $\alpha'_1 \cdot \beta'_1 > 0$, since v lies outside the segment $[f'_2, e'_2]$. This Lagrangian will intersect the subspace S_1 spanned by e_1, f_1 at the point $\alpha'_1(e_1 + \frac{1}{6} f_1) + \beta'_1 f_1$, which can be checked directly from the formulas for v, e'_1, f'_1. But this intersection point has both coordinates positive with respect to e_1, f_1, and since v also has both coordinates positive with respect to e_2, f_2, it follows that this Lagrangian is not in $\mathcal{S}$ (see §5.1.12). $\square$

5.2.7. Corollary (Cutting along crooked surfaces)**.** *Suppose that two crooked surfaces $\mathcal{WS}, \mathcal{WS}'$ are either in the configuration of Proposition 5.2.3, i.e. disjoint, or in one of the configurations in Proposition 5.2.6.*

Then $\mathcal{WS}'$ is entirely contained in one component of $\mathrm{LGr}(V) \setminus \mathcal{WS}$ in the first case, or contained in a component except for a set of photons along which in intersects $\mathcal{WS}$ in the second case.

5.3. Domain of discontinuity

5.3.1. Setup. We now apply the preceding formalism of crooked surfaces to analyze the domains of discontinuity for our groups Γ_N from Theorem 3.2.5, using the cones constructed in Section 3.

Let $I := \{0, 0', \ldots, (N-1), (N-1)'\}$ denote the indexing set for the cones. Given $i \in I$, we have a cone $\mathcal{C}_i \subset \mathbb{P}(V)$, and an associated reflection $A_i \in \mathbf{GSp}(V)$. In the Lagrangian Grassmannian we then obtain a crooked surface $\mathcal{WS}_i \subset \mathrm{LGr}$, and its complement decomposes into two open sets

$$\mathrm{LGr}(V) \setminus \mathcal{WS}_i = \mathcal{L}_{i,s} \coprod \mathcal{L}_{i,b}$$

where the "small" open set $\mathcal{L}_{i,s}$ is associated to the component determined by the cone $\mathcal{C}_i$, and the "big" open set $\mathcal{L}_{i,b}$ is associated to the component determined by the cone $A_i \cdot \mathcal{C}_i$. The reflection A_i preserves $\mathcal{WS}_i$ as a set, and exchanges the two components $\mathcal{L}_{i,\bullet}$. We will denote by $\overline{\mathcal{L}}$ the closure of a component (so $\overline{\mathcal{L}} = \mathcal{L} \coprod \mathcal{WS}$).

5.3.2. Left and right adjacency. Recall that the indexing set I for the cones is cyclically ordered. Let then $r(i)$, resp. $l(i)$, denote the right, resp. left, neighbors of the element i. We will also use the composition of reflections:

$$T_{i,r} := A_i A_{r(i)} \quad T_{i,l} := A_i A_{l(i)}$$

which for adjacent vertices satisfy $T_{i,r} \cdot T_{r(i),l} = 1$ and $T_{i,l} \cdot T_{l(i),r} = 1$. Note that each $T_{i,l/r}$ is a unipotent transformation taking the cone $\mathcal{C}_i$ to itself, and one of the matrices is a rank 1 unipotent while the other is maximally unipotent.

5.3.3. Finite approximations to limit set and domain of discontinuity. We can now combine the calculations with containments of cones from Theorem 3.2.5 with the disjointness/touching criteria from Proposition 5.2.3 and Proposition 5.2.6. It follows that when $i \neq j$, we have that $\mathcal{WS}_j \subset \overline{\mathcal{L}_{i,b}}$ and more generally $\overline{\mathcal{L}_{j,s}} \subset \overline{\mathcal{L}_{i,b}}$.

Let us define

$$\Lambda_1 := \bigcup_{i\in I} \overline{\mathcal{L}_{i,s}} \quad \text{and} \quad \Omega_1 := \bigcap_{i\in I} \mathcal{L}_{i,b} = \mathrm{LGr}(V) \setminus \Lambda_1$$

These provide a first approximation to the domain of discontinuity Ω and limit set Λ. We can define Λ_n and $\Omega_n := \mathrm{LGr}(V) \setminus \Lambda_n$ recursively, or in a more direct manner:

$$\begin{aligned} \Lambda_n &:= \{x \in \mathrm{LGr}(V)\colon \exists i_1, \ldots, i_n \in I \text{ s.t. } i_l \neq i_{l+1} \\ &\qquad \text{and } x_{i_1} \in \overline{\mathcal{L}_{i_1,s}} \text{ s.t. } x = A_{i_n} \cdots A_{i_2} \cdot x_{i_1}\} \\ \Omega_n &:= \{x \in \mathrm{LGr}(V)\colon \forall i_1, \ldots, i_n \in I \text{ s.t. } i_l \neq i_{l+1} \\ &\qquad \text{we have that } A_{i_2} \cdots A_{i_n} \cdot x \notin \overline{\mathcal{L}_{i_1,s}}\} \end{aligned} \tag{5.3.4}$$

It is immediate from the definitions that $\Omega_n = \mathrm{LGr}(V) \setminus \Lambda_n$, and that Λ_n is closed (resp. Ω_n is open). Let us point out that the sequence $i_1, \ldots, i_n$ which certifies that $x \in \Lambda_n$ need not be uniquely associated to x.

We also have that $\Lambda_{n+1} \subset \Lambda_n$ since if $x \in \Lambda_{n+1}$ with $x = A_{i_{n+1}} \cdots A_{i_2} \cdot x_{i_1}$ then we can also use $x_{i_2} := A_{i_2} x_{i_1}$ and the last n terms of the sequence, to see that $x \in \Lambda_n$, since $A_{i_2} \cdot \overline{\mathcal{L}_{i_1,s}} \subset \overline{\mathcal{L}_{i_2,s}}$. Similarly note that $\Omega_{n+1} \supset \Omega_n$, since if $x \notin \Omega_{n+1}$ then there exists a sequence $i_1, \ldots, i_{n+1}$ with $A_{i_2} A_{i_3} \cdots A_{i_{n+1}} x \in \overline{\mathcal{L}_{i_1,s}}$, but then $A_{i_3} \cdots A_{i_{n+1}} x \in A_{i_2} \overline{\mathcal{L}_{i_1,s}} \subset \overline{\mathcal{L}_{i_2,s}}$ showing that $x \notin \Omega_n$ either.

5.3.5. A preliminary domain of discontinuity. We can define now the sets

$$\Lambda^\circ := \bigcap_{n\geq 1} \Lambda_n \text{ and its complement } \Omega^\circ := \bigcup_{n\geq 1} \Omega_n.$$

By construction Λ° is closed and Ω° is open, and both sets are Γ_N-invariant. We will see in §5.3.8 below that Ω° can be slightly enlarged to a bigger Γ_N-invariant set, while its complement Λ° can be slightly enlarged.

5.3.6. Boundary of the fundamental domain. If we denote by $\overline{\Omega}_1^{rel} \subset \Omega^\circ$ the relative closure of the first domain Ω_1, then it is immediate to check from the properties of the action, and the definitions, that the Γ_N-orbit of any point intersects $\overline{\Omega}_1^{rel}$.

Let us further analyze the boundary of this fundamental domain. We have the following set-theoretic calculations:

$$\Omega_2 \setminus \left(\Omega_1 \cup \bigcup_{i\in I} A_i \cdot \Omega_1\right) = \Omega_1 \cap \left(\Lambda_1 \cap \bigcap_{i\in I} A_i \cdot \Lambda_1\right)$$
$$= \Omega_2 \cap \bigcap_{i\in I} \mathcal{WS}_i.$$

Consider now a point $x \in \Omega_2 \cap \mathcal{WS}_{i_0}$. The assumption $x \in \Omega_2$ is equivalent to the statement that for any $i_1 \neq i_2$ we have that $A_{i_2}x \notin \overline{\mathcal{L}_{i_1,s}}$. Observe that if i_2 is not adjacent to i_0 then this is automatic since the reflection A_{i_2} will map $\mathcal{L}_{i_0,s}$ strictly inside $\mathcal{L}_{i_2,s}$, and so the same will remain true of the boundary $\mathcal{WS}_{i_0}$.

So we have to consider the cases $i_2 \in \{r(i_0), l(i_0)\}$. By an analogous reasoning, if $i_1 \neq i_0$ then we have $A_{i_2}\mathcal{WS}_{i_0} \cap \overline{\mathcal{L}_{i_1,s}} = \emptyset$, so we have to consider only the case $i_1 = i_0$. A point $x \in \Omega_2 \cap \mathcal{WS}_{i_0}$ is characterized by

$$A_{r(i_0)}x \notin \overline{\mathcal{L}_{i_0,s}} \text{ and } A_{l(i_0)}x \notin \overline{\mathcal{L}_{i_0,s}}$$

which, by applying A_{i_0} to both sides, and using the notation from §5.3.2, is equivalent to

$$T_{i_0,r}x \notin \overline{\mathcal{L}_{i_0,b}} \text{ and } T_{i_0,l}x \notin \overline{\mathcal{L}_{i_0,b}}$$

In other words we have

$$\Omega_2 \cap \mathcal{WS}_{i_0} = \mathcal{WS}_{i_0} \setminus (T_{i_0,r} \cdot \mathcal{WS}_{i_0} \cup T_{i_0,l} \cdot \mathcal{WS}_{i_0})$$

In other words, we must eliminate the intersections of the original crooked surface with its translates by two unipotent transformations. These are precisely the sets described in Proposition 5.2.6: one intersection is along a full wing of the surfaces, while another is along a single photon.

5.3.7. Action of reflections on a wing. It follows from the previous analysis that the sets Λ_n will contain full wings of adjacent crooked surfaces, along which the sources "touch" in the sense of Proposition 5.2.6. Let us analyze now the dynamics of the two reflections which fix, as a set, the particular wing. Up to conjugacy, the model is that of the group generated by the matrices A, B from §2.2.4. Their product $AB = (A-B)B+1$ is a rank 1 unipotent matrix, since $A-B$ is visibly a rank 1 matrix.

Recall also that the cone is given in Eqn. (3.4.6) and its column vectors are (up to scaling) what we called e_1, f_1, e_2, f_2. Then the geometry is as follows. The group generated by A, B preserves as a set each photon $\phi(v)$ for $v = \alpha_1 e_1 + \alpha_2 e_2$. The two reflections fix the

Lagrangian $L_{e_1e_2}$, which lies on each of the photons in question. Identifying $\phi(v) = \mathbb{P}(v^\perp/v) \cong \mathbb{P}^1(\mathbb{R})$, and removing "the point at infinity" $L_{e_1e_2}$, the action of A, B then becomes that of two Euclidean reflections on $\mathbb{R}$. Under increasingly longer words in A, B the orbit of a point approaches the point at infinity $L_{e_1e_2}$.

The above description holds except for a photon $\phi(v_0)$, where v_0 is the image of $AB - \mathbf{1}$. The action of the group generated by A, B is trivial on this photon, and the vector v_0 is the "attractor" for the projective action of large powers of AB. Let us call $\phi(v_0) \subset \mathcal{W}_e$ the "attractor photon" on the corresponding e-wing.

5.3.8. Enlarging the domain of discontinuity. We can now enlarge our open set Ω° to a larger domain of discontinuity, as follows. Enlarge Ω_2 by adding, for each index $i \in I$, the complement in the e-wing of the "attractor photon". Then, take the image of Ω_2 under the group Γ_N and call the resulting set Ω, with complement Λ.

This construction is equivalent to removing from the limit set Λ_2 the e-wings, except for the attractor photons, and then taking successive images and intersecting as in §5.3.3.

Let us finally remark that the limit set Λ intersects each winged surface in two photons only, namely the f-vertex photon which is the attractor for the maximally unipotent matrix, and another photon attractor for the rank 1 unipotent transformation on the e-wing.

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