Ergodic properties of Rauzy–Veech inductions

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Abstract

We introduce a unified description of Rauzy–Veech induction on interval exchanges and linear involutions with or without flips using simplicial systems. This enables us to give a new straightforward and common proof of the existence and uniqueness (corresponding to Masur–Veech measure) of the measure of maximal entropy for Teichmüller flow on abelian and quadratic differentials strata. Other dynamical consequences imply a central limit theorem generalizing to Bufetov's as well as a key ingredient for Avila–Gouezel–Yoccoz and Avila–Resende proof of exponential mixing of Teichmüller flow.

For interval exchanges and linear involutions with flips we obtain the existence of a periodic subinterval for almost every parameters as well as an upper bound on the Hausdorff dimension of the complementary set of such parameters. This strengthens the results of Nogueira, Danthony–Nogueira and Skripchenko–Troubetzkoy and introduce an analog of Veech flow in this cases which should conjecturally be conjugated to Teichmüller flow on (non-generic) measured foliation without closed one-sided leaves.

Contents

1 Introduction

In the study of dynamics of measured foliations on surfaces, a natural idea — going back at least to Poincaré — when the objects are oriented is to consider its first return map to a transverse interval. Such maps, as they preserve a measure, are locally translations with finitely many domains of continuity. It thus produces a map which divides the interval in a finite number of pieces and permutes them hence such maps bear the name of interval exchanges transformations.

Another seminal idea to understand these objects, which boomed in the 70's and the work of Feigenbaum, is renormalization. It was used in the early 80's by Masur and Veech simultaneously and independently to prove the first ergodic properties of measure foliations. Veech used a renormalization directly on the interval exchange transformation, called Rauzy–Veech induction. It consists in associating to an interval exchange map its first return map a well chosen subinterval, such that the new map has the same number of interval domains. Thus creating a sequence of IET from an initial one. Renormalization idea, uses the fact that the dynamics of one interval exchange can be better understood by describing this sequence.

In another perspective, this renormalization can be interpreted has a first return map of a deformation flow on the moduli space of measured foliations on surfaces and their natural extension, translation surfaces. This flow is commonly called Teichmüller flow. Masur and Veech introduced an invariant finite measure equivalent to Lebesgue measure and showed its ergodicity with respect to the renormalization map.

Many ergodic results in the field have used this technique ever since. Let us mention here a very partial list : The unique ergodicity of almost every measured foliation [Ker85], Central limit theorem [Buf06], Exponential mixing [AGY06], Existence and uniqueness of a measure of maximal entropy [BG11], Hausdorff codimension of non-uniquely ergodic measured foliations [CM20].

Orientable surfaces A rule of thumb in the field states that the ergodic properties of Teichmüller flow on the moduli space of translation surfaces can be transposed to the Teichmüller flow on the moduli space of *half*-translations surfaces, corresponding in our setting to non-orientable foliations on orientable surfaces.

The proofs usually turn out to have similar schemes but with many specific technicalities appearing in the way.

The analog of interval exchange transformation in this context was introduced by Danthony–Nogueira in [DN90]. They coined it Linear Involution, since it is not a map, but involves an involution switching two intervals. This was used for instance by Avila–Resende in [AR12] to prove Exponential mixing in this case.

Nonetheless the more technical nature of the proofs together with the fact that it does not seem to produce original behaviours for many of these results makes the subject to be under consideration. Several of the ergodic properties for orientable measured foliations have never been proved in the case of non-orientable ones.

In the following work, we introduced a unifying formalism, taken from a companion work [Fou24a], which enable us to give common proofs for both these cases as well as new insights on each of them. This part of a more general work which introduces a formalism and techniques to study such maps with a probabilistic approach by random walk ideas. As consequence, we are able to revisit Masur–Veech result, proving exponential tail property for Teichmüller flow in these two settings.

In particular we get a generalization of Bufetov–Gurevich results.

Theorem I. *The Masur–Veech measure is the unique measure of maximal entropy for the Teichmüller flow on strata of abelian and quadratic differentials with area 1. Its entropy is equal to $d = 2g + n - 1$, i.e. the complex dimension of the stratum before normalization.*

The formula for the entropy was claimed by Kontsevich in [Kon97] up to adapting Pesin theory in this context.

Another consequence is a Central Limit theorem for non-orientable foliations (Theorem 4.16) generalizing Bufetov theorem [Buf06] on orientable foliations. Moreover, this work also covers in a common framework the proof of exponential tail property in both this cases, giving a common framework for proving this key ingredient of exponential mixing of the Teichmüller flow.

Non-orientable surfaces It was noticed by Nogueira [Nog89] and latter by Danthony–Nogueira [DN90] that the generic dynamics of measured foliations on non-orientable surfaces is of completely different nature as in the orientable case. Far from being uniquely ergodic, they are expected to contain a Möbius strip, *i.e.* a union of non-orientable closed leaves of positive measure.

Their proof relies on another generalization of interval exchange, introducing flips, intervals on which the map reverses orientation, locally $-id$ composed with a translation. It is also possible to generalize it to linear involution with flips. All these objects will be defined precisely in Section 2.

More recently, Skripchenko–Troubetzkoy [ST18] gave a proof that the set of parameters for which interval exchanges with flips do not contain a periodic orbit have Hausdorff dimension strictly smaller that the dimension of its ambiant space. These three works all rely on a conditioning argument on this subspace that they intend to prove is of zero measure. The definition of this conditioning thus comes with an important gap. We propose a way to fill in this gap and generalize the result to linear involutions.

Theorem II. *Consider the set of length parameter for interval exchanges (resp. linear involution) with flips. The subset of interval exchanges (resp. linear involutions) which do not contain a Möbius strip is of Hausdorff dimension strictly smaller than the dimension of the total set.*

2 Definitions

2.1 Win-lose induction

Let $G = (V, E)$ denote a graph labeled on an alphabet $\mathcal{A}$ by a function $l : E \to \mathcal{A}$ such that all vertex $v \in V$ has either zero or two outgoing edges with distinct labels.. Moreover, for every $v \in V$, the restriction of l to E_v, the set of edges going out of v, is assumed to be injective.

Let V^0 be the set of vertices in V with no outgoing edges. A vertex v in $V \setminus V^0$ has by assumption two outgoing edges e, f respectively labeled by $\alpha, \beta \in \mathcal{A}$. The subcones

$$\mathcal{K}^e := \left\{ \lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha < \lambda_\beta \right\} \quad \text{and} \quad \mathcal{K}^f := \left\{ \lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\beta < \lambda_\alpha \right\}$$

form a partition of $\mathbb{R}_+^{\mathcal{A}}$ where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$. They have the same boundary set thus depending only on the vertex v

$$\mathcal{H}^v := \left\{ \lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha = \lambda_\beta \right\}.$$

Additionally, we associate matrices

$$M_e := \mathrm{Id} + E_{\beta,\alpha} \quad \text{and} \quad M_f := \mathrm{Id} + E_{\alpha,\beta}.$$

Where $E_{a,b}$ is the elementary matrix with coefficient 1 at row a and column b. Such that $\mathcal{K}^e = M_e \cdot \mathbb{R}_+^{\mathcal{A}}$ and $\mathcal{K}^f = M_f \cdot \mathbb{R}_+^{\mathcal{A}}$. Hence it is natural to define

$$\Theta_e : \left\{ \begin{array}{ccc} \mathcal{K}^e & \to & \mathbb{R}_+^{\mathcal{A}} \\ \lambda & \mapsto & M_e^{-1}\lambda \end{array} \right. .$$

The *win-lose induction* associated to the graph G is the map

$$\Theta : \left(V \setminus V^0\right) \times \mathbb{R}_+^{\mathcal{A}} \to V \times \mathbb{R}_+^{\mathcal{A}}$$

defined for every edge e from vertices v to v' and all $\lambda \in \mathcal{K}^e$ by $\Theta(v, \lambda) = (v', \Theta_e(\lambda))$.

Remark 2.1. *The map is only defined on*

$$\bigsqcup_{v \in V \setminus V^0} \mathbb{R}_+^{\mathcal{A}} \setminus \mathcal{H}^v$$

but we make this abuse of notation for clarity, since these hyperplanes will not play a role in the Lebesgue generic dynamical behaviour nor the Hausdorff dimensions we will estimate.

Consider a vertex v with two or more outgoing edges and a parameter $\lambda \in \mathbb{R}_+^{\mathcal{A}}$. In analogy with Rauzy–Veech induction (for an introduction, refer to [Yoc10]), we call the edge e such that $\lambda \in \mathcal{K}^e$ the **loser**. Conversely, the labels of any other edge e' in E_v is called a **winner**, and we say it wins against e. We sometimes say a label wins or loses when there is no ambiguity to which edge they correspond.

The map Θ can be characterized as follows: it compares the coordinates of all edges emanating from a given vertex v on the vector and subtracts the smallest coordinate from the others, effectively subtracting the losing coordinate from the winning ones.

Remark 2.2. *In the following, we denote an edge by its label when there is no ambiguity. Using for instance $\mathcal{K}^\alpha$ instead of $\mathcal{K}^e$.*

Let us consider the projectivization relation $x \sim \lambda x$ satisfied for all $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}_+^{\mathcal{A}}$. We denote by $\Delta := \mathbb{R}_+^{\mathcal{A}}/\sim$ the simplex of dimension $|\mathcal{A}| - 1$ and, for each $v \in V$, by $\{\Delta_e\}_{e \in E_v}$ its induced partition by $\{\mathcal{K}^e\}_{e \in E_v}$. The maps Θ_e can be quotiented by this relation and we denote the induced map by $T_e : \Delta_e \to \Delta$. Similarly, Θ induces a map on space $\Delta(G) := V \times \Delta$ denoted by $T : \Delta(G) \to \Delta(G)$.

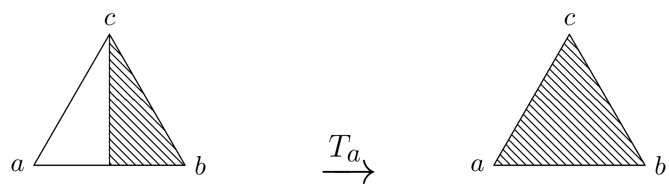


Figure 1: Action of T_a on Δ_a

Remark 2.3. *Classically (see [Yoc10]), Rauzy–Veech induction is represented with its Rauzy diagram, a graph with vertices labeled with the corresponding interval exchange permutation which edges are labeled by* top *of* bottom *depending on which interval* wins *in the induction and points to the corresponding new permutation. As a win-lose induction, we prefer to label the edges by the* losing label. *In Figure 2, we represent Rauzy diagram for a 3-IET with the labeling of its corresponding win-lose induction.*

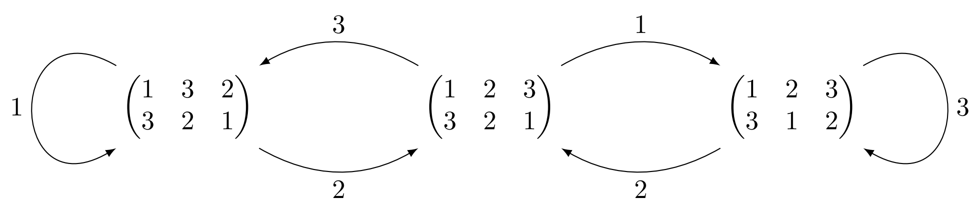


Figure 2: Rauzy diagram for 3-IET.

2.1.1 Linear subspace restriction

Let $\{\phi_v\}_{v \in V}$ be a family of preserved linear forms, *i.e.* satisfying $\phi_v(x) = \phi_{v'}(x')$ for all $v, v' \in V$ and $x, x' \in \Delta$ such that $T(v, x) = (v', x')$.

Let us denote by Δ_v^ϕ the convex polytope $P(\mathbb{R}_+^{\mathcal{A}} \cap \ker \phi_v)$ and V^ϕ the set of vertices v such that Δ_v^ϕ has same dimension as $\ker \phi$. Together with edges between vertices in V^ϕ this defines a subgraph G^ϕ of G such that a path associated to parameters in Δ_v^ϕ remains in G^ϕ. In particular, it induces a restricted map on

$$\Delta(G^\phi) := \bigsqcup_{v \in V^\phi} \Delta_v^\phi$$

denoted by $T^\phi : \Delta(G^\phi) \to \Delta(G^\phi)$.

Such restrictions will be of main interest here to study dynamics of linear involutions. Indeed, for linear involutions, there exists a generalization of Rauzy–Veech induction which is constrained by the condition that top and bottom total lengths must coincide. This is illustrated in the case of a linear involution on 3 intervals in Figure 3. The dotted line corresponds to the splitting of the win-lose induction and the thick line is the orthogonal subspace defined by lengths equality.

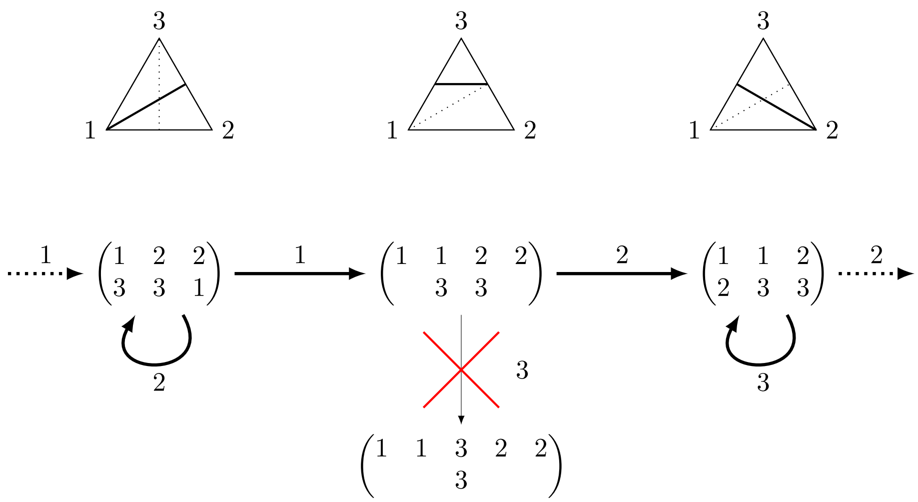


Figure 3: Rauzy diagram for linear involution on 3 intervals.

A degenerate case can happen where the length condition implies existence of a saddle connection. Consider for instance the matching

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix}.$$

In this case, the length condition implies equality of length for the ending labels. In other words, $\ker \phi_v$ coincides with $\mathcal{H}^v$ and this vertex has no outgoing edges in G^ϕ.

Remark 2.4. *The linear forms induce a Lebesgue measure on each* $\ker \phi_v$ *which we denote by* Leb^ϕ*. On extremal strongly connected components of* G^ϕ *(for which there is no edges going out of the component), it comes with a specific property denoted by* $(\mathrm{Leb}^\phi_\sim)$ *in [Fou24a]. When it is not checked one only has* $(\mathrm{Leb}^\phi_\leq)$ *as for subgraph restrictions in the next subsection.*

2.1.2 Subgraph restriction

Let $F = \left(V^F, E^F\right)$ be a subgraph of a win-lose induction base graph G. Consider the subset of parameters in $\Delta(G)$ which remain in F for n steps

$$\Delta^n(F) = \bigsqcup_{v \in V^F} \left(\bigcup_{\substack{\gamma \in \Pi(v) \\ |\gamma| = n}} M_\gamma \Delta_{\gamma \cdot v} \right)$$

where $\Pi(v)$ is the set of finite paths in G starting at v and $|\gamma|$ denotes the length of path γ. This also induces a restricted induction on the space of parameters remaining indefinitely in F, $\Delta(F) = \bigcap_{n \in \mathbb{N}} \Delta^n(F)$. Denoted by $T^F : \Delta(F) \to \Delta(F)$.

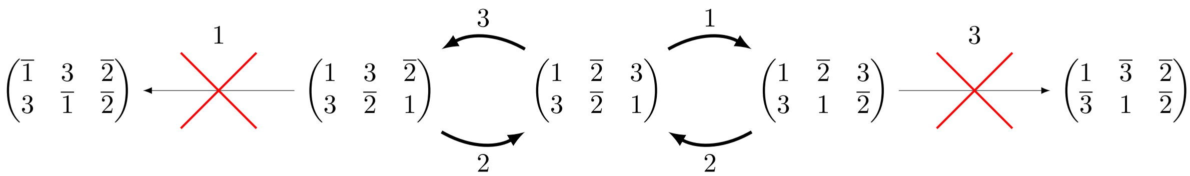
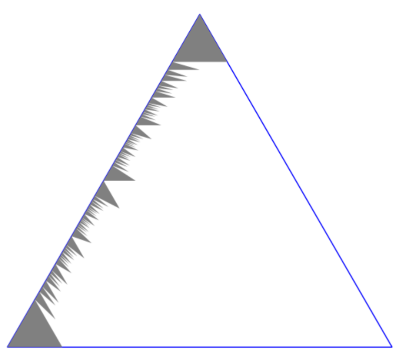
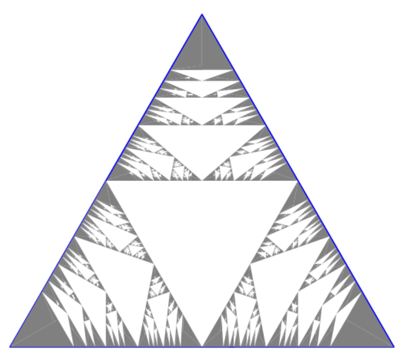


Figure 4: Rauzy diagram for 3-IET with flip.

For flipped interval exchanges and linear involutions, we will be interested into parameters for which the induction does not meet permutations where the end labels of top and bottom intervals differ. An example on 3 intervals is given by the subgraph with vertices on the top line and thick edges in Figure 4. This converge to an empty for such simple combinatorics, but non-trivial fractal sets appear for higher dimension or more general constructions (such as Rauzy gasket studied in [Fou24b]).



(a) Parameters remaining in the subgraph of Figure 4 for 12 steps.



(b) Another such set converging to a non-trivial fractal.

The two examples above and their generalization will be the heart of Section 3.

2.1.3 Suspension

Let us define the *roof function* for (almost) all $x \in \Delta(G)$ as follows. Let e be the edge from vertex v to v' such that $x \in \{v\} \times M_e \Delta_{v'}$, we set

$$r(x) = -\log\left(\frac{|M_e^{-1} x|}{|x|}\right). \tag{1}$$

Define the *suspension space* $\Delta(G)_r := (\Delta(G) \times \mathbb{R}) / \sim$, where for all $(x,t) \in \Delta(G) \times \mathbb{R}$ we have the equivalence $(x,t) \sim (Tx, t + r(x))$. The associated *suspension semi-flow* is defined on $\Delta(G)_r$, for all $t \geq 0$, by

$$\phi_t : (x,s) \to (x, s+t).$$

Notice that this flow is defined such that the first return map to the section $\Delta(G) \times \{0\}$ is T and its return time is r.

Denote by $\mathcal{M}_{T,r}$ the set of T-invariant Borel probability measures with $\mu(r) := \int_{\Delta(G)} r d\mu < +\infty$. Every ϕ-invariant Borel probability measure $\widetilde{\mu}$ on $\Delta(G)_r$ can be decomposed as a product of a measure $\mu \in \mathcal{M}_{T,r}$ and the Lebesgue measure on fibers. Namely,

$$\widetilde{\mu}_r = (\mu(r))^{-1} (\mu \times \text{Leb})_{|\Delta(G)_r}.$$

The Kolmogorov–Sinai entropy of the flow for this measure is written $h(\phi, \widetilde{\mu})$ and satisfies Abramov's formula

$$h(\phi, \widetilde{\mu}) = \frac{h(T, \mu)}{\mu(r)}$$

where $h(T, \mu)$ is the Kolmogorov–Sinai entropy for T. In this setting the topological entropy can be defined as

$$h_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}_{T,r}} h(\phi, \widetilde{\mu}_r).$$

The induced measure $\widetilde{\mu}_r$ for $\mu \in \mathcal{M}_{T,r}$ at which this supremum is achieved (and by extension μ itself) is referred to as a *measure of maximal entropy*.

Again the suspension can be restricted to subgraph on spaces $\Delta(G^\phi)_r := (\Delta(G^\phi) \times \mathbb{R})/\sim$ and $\Delta(F)_r := (\Delta(F) \times \mathbb{R})/\sim$ respectively.

2.2 Non-degenerating properties

In a companion work, we have developped criteria on graphs and subgraphs for the win-lose induction which implies many ergodic properties of the map and its suspension flow. We first present the criterion on a full graph and then for a subgraph.

2.2.1 On full graphs

Assume that, on a non-trivial subset of labels $\mathcal{L} \subset \mathcal{A}$, the parameter in $\mathbb{R}_+^{\mathcal{A}}$ have coordinates in $\mathcal{L}$ infinitely smaller than others. At a vertex with at one outgoing edge labeled in $\mathcal{L}$, any other edge labeled outside of $\mathcal{L}$ must win. Hence, the map T will remain in a subgraph in which we remove such edges not labeled in $\mathcal{L}$.

This motivates the introduction of the *degenerate subgraph* $G^{\mathcal{L}}$ having the same set of vertices V as G but for which we remove edges along which a letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$. For a vertex $v \in V$ in $G^{\mathcal{L}}$, the set of outgoing edges is defined as follows.

- If $l(E_v) \cap \mathcal{L} \neq \emptyset$

$$E_v^{\mathcal{L}} = \{e \in E_v \mid l(e) \in \mathcal{L}\}.$$

- Otherwise

$$E_v^{\mathcal{L}} = E_v.$$

Definition 2.5 (Non-degenerating graph)**.** *We say that the base graph of a vector memory random walk is* non-degenerating *if it is strongly connected and, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and all vertices v in a strongly connected component $\mathscr{C}$ of $G^{\mathcal{L}}$, one of the following properties holds:*

1. *There is a path from v in G labeled in $\mathcal{L}$ leaving $\mathscr{C}$.*

2. $|l(E_v) \cap \mathcal{L}| \leq 1$.

In plain words: from any vertex, no letter in $\mathcal{L}$ can win against another letter in $\mathcal{L}$ in any strongly connected component of $G^{\mathcal{L}}$ except if there is a path labeled in $\mathcal{L}$ leaving the component.

It is easy to check that this property is satisfied by a Rauzy diagrams associated to an irreducible IET (see Proposition 2.15 in [Fou24a]). This implies many ergodic properties for Rauzy–Veech induction in this case, as a consequence of the following theorem in [Fou24a].

Theorem A. *Every non-degenerating win-lose induction has a unique invariant measure equivalent to Lebesgue measure and it induces the unique invariant probability measure of maximal entropy for the (semi-)flow on its canonical suspension.*

Moreover, the entropy of the canonical suspension flow is equal to $|\mathcal{A}|$.

2.2.2 On subgraphs

In certain cases, such as those illustrated in Figure 3 and Figure 4, it becomes necessary to consider a subgraph F of a graph G that defines a win-lose induction. Within such subgraphs, we often encounter vertices that have a unique outgoing edge.

From a dynamical point of view, these vertices can be bypassed in the orbit of the win-lose induction, until we encounter a *branching vertex* — that is, a vertex with multiple outgoing edges. This observation motivates the introduction of a *factorization* of the graph.

More precisely, we aim to associate to each *degenerating* subset of labels a corresponding subset of vertices, on which we define an accelerated version of the induction, distinct from the one induced on the factor graph.

To ensure that this acceleration process remains well-defined and finite, we first impose a structural condition on these families. Furthermore, since we wish to keep track of the labels in the degenerating subset, we require that each such label intervenes only once at each step of the accelerated induction.

Definition 2.6 (Filling factoring family)**.** *Consider, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, a subset $\widetilde{V}_{\mathcal{L}}$ of vertices of F such that every loop in this subgraph contains a vertex in $\widetilde{V}_{\mathcal{L}}$. Let $\widetilde{E}^{\mathcal{L}}$ be the set of finite path in F such that*

- *Its start and end vertex belong to $\widetilde{V}_{\mathcal{L}}$ and no other visited vertices do.*
- *Along this path, no letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$.*

These paths are called $\mathcal{L}$-factor paths.

We say the collection $\{\widetilde{V}_{\mathcal{L}}\}$ is a filling factoring family *if every $\mathcal{L}$-factor path γ visits at most one branching vertex v satisfying $l(E_v) \cap \mathcal{L} \neq \emptyset$ and which is not the end vertex of γ. We call v the $\mathcal{L}$-branching vertex of γ when it exists.*

Notice that the degenerate subgraph $F_{\mathcal{L}}$ is composed of edges appearing in paths of $\widetilde{E}^{\mathcal{L}}$.

Remark 2.7. *The condition on loops implies that $\widetilde{E}^{\mathcal{L}}$ is finite.*

Notice that we say a letter α wins against another letter β along a *path* $\gamma = e_1 \dots e_n$ if α wins against β along an edge (in G) or α wins against δ which has won against β before, etc. In other term, if the matrix $M_{e_1} \dots M_{e_n}$ is positive at (α, β).

A generalization of the non-degenerating criterion is then defined for such families.

Definition 2.8 (Non-degenerating family)**.** *A filling factoring family $\{\widetilde{V}_{\mathcal{L}}\}_{\mathcal{L}}$ forms a* non-degenerating factorization *of the subgraph F if for every $\emptyset \subset \mathcal{L} \subset \mathcal{A}$ and $\mathcal{L}$-factor path $\gamma \in \widetilde{E}^{\mathcal{L}}$ contained in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$ one of these properties is true.*

1. *There exists a path in F starting at a vertex of γ which leaves $\mathscr{C}$ and such that each edge in the path based at a branching vertex of F is labeled in $\mathcal{L}$.*
2. *All edges in γ are labeled by letters not in $\mathcal{L}$.*
3. *The path has a $\mathcal{L}$-branching vertex with a unique outgoing edge labeled in $\mathcal{L}$ and each label winning against a letter in $\mathcal{L}$ along γ is in $l(E_v^F) \setminus \{\alpha\}$.*
4. *The path does not meet any $\mathcal{L}$-branching vertex, it has a unique winning label β and at least one losing label not in $\mathcal{L}$.*

Notice in particular similarities of 1 and 3 in this definition with 1 and 2 in Definition 2.5.
A generalization of Theorem A under hypothesis ($\mathrm{Leb}^{\phi}_{\sim}$) was also proved in [Fou24a]. Here, we will just use that this hypothesis is satisfied on extremal strongly connected components of G^{ϕ}. It corresponds to the (strong) irreducibility property for linear involutions defined by Boissy–Lanneau [BL09], see Section 3.3.

Theorem A'. *Consider a win-lose graph endowed with an invariant family of linear forms. Assume there is an extremal strongly connected component of the induced subgraph G^{ϕ} which is non-degenerating. Then the map T^{ϕ} restricted to this component has a unique invariant measure equivalent to Lebesgue measure and it induces the unique invariant probability measure of maximal entropy for the (semi-)flow on its canonical suspension.*

Moreover, the entropy of the canonical suspension flow is equal to the dimension of $\ker \phi_v$, in previous notations $|\mathcal{A}| - 1$.

For other non-extremal strongly connected components F which can appear in the flipped cases, one only has a weaker property ($\mathrm{Leb}^{\phi}_{\leq}$). Nonetheless the parameter space $\Delta(F)$ has holes (coming from paths going out of the component) and we can bound its Hausdorff dimension.

Theorem B. *Let F by a non-extremal strongly connected component of a win-lose graph G and assume it is non-degenerating. Its canonical suspension flow admits a unique measure of maximal entropy, with entropy $0 < h < |\mathcal{A}|$, and the Hausdorff dimension*

$$\dim_H \Delta(F) \leq |\mathcal{A}| - 2 + \frac{h}{|\mathcal{A}|} < \dim \Delta(G).$$

One also has a version of the theorem for subgraphs taken in the linear subspace restriction G^{ϕ} of a win-lose graph with an invariant family of linear forms $\{\phi_v\}_{v \in V}$.

Theorem B'. *Let F by a non-extremal strongly connected component of G^{ϕ} and assume it is non-degenerating. Its canonical suspension flow admits a unique measure of maximal entropy, with entropy $0 < h < |\mathcal{A}| - 1$, and the Hausdorff dimension*

$$\dim_H \Delta(F) \leq |\mathcal{A}| - 3 + \frac{h}{|\mathcal{A}| - 1} < \dim \Delta(G^{\phi}).$$

2.3 Central Limit Theorem

On $\Delta(F)$, there is a natural metric given by Hilbert metric on the simplices $\widehat{\Delta}^G$. And one can associate a product metric on $\widehat{\Delta}_r^G$ with the euclidien metric in fibers.

For $\alpha > 0$, let us denote by $H^\alpha(\widehat{\Delta}_r^G)$ the space of α-Hölder functions for this metric. The canonical suspension flow in these cases also satisfies the following.

Theorem C. *Let $p > 2$ and let $f \in H^\alpha(\widehat{\Delta}_r^G) \cap L^p(\widehat{\Delta}_r^G, \mu_r)$ satisfy $\int f \, \mathrm{d}\mu_r = 0$. Assume that there does not exist $\widetilde{f} \in L^2(\widehat{\Delta}_r^G, \mu_r)$ differentiable in the direction of the suspension such that $f = X_t \widetilde{f}$ where X_t is the Lie derivative in that direction. Then there is a positive constant σ_f such that*

$$\frac{1}{\sqrt{|L|}} \int_0^L f \circ \Phi_t \, dt \xrightarrow{d} \mathcal{N}(0, \sigma_f) \quad \text{as } |L| \to \infty.$$

Where the convergence is in distribution to a normal law of variance σ_f.

3 Rauzy–Veech inductions

3.1 Definitions

For w a finite word in the finite alphabet $\mathcal{A}$ we denote by $|w|_x$ the number of occurrences of the letter $x \in \mathcal{A}$ in the word.

Definition 3.1 (Signed matching). *A signed matching $m = (\nu, \omega, \epsilon)$ is given by two words ν and ω in an alphabet $\mathcal{A}$ which satisfy, for all $x \in \mathcal{A}$,*

$$|\nu|_x + |\omega|_x = 2$$

and a sign map

$$\epsilon : \mathcal{A} \mapsto \{\pm 1\}.$$

We say a letter $\alpha \in \mathcal{A}$ is double if $|\nu|_\alpha$ or $|\omega|_\alpha$ is 2. Otherwise, we say the letter is simple. For a length vector $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ we denote the length of w by

$$\lambda(w) := \sum_{i=1}^{|w|} \lambda_{w_i}.$$

Definition 3.2 (Linear involution). *Let $m = (\nu, \omega, \epsilon)$ be a signed matching and a length vector $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ such that $\lambda(\nu) = \lambda(\omega)$. We call the couple (m, λ) a linear involution.*

Let us follow the definition in [DN90]. Notice that the sign function ϵ describe the horizontal inversion. Let us define a function δ which describes the vertical inversion for all $x \in \mathcal{A}$,

$$\delta(x) = \begin{cases} 1 & \text{if } |\nu|_x = 1 \\ -1 & \text{otherwise} \end{cases}.$$

Definition 3.3. *Consider the set of flipped intervals, $F = \{x \in \mathcal{A} \mid \epsilon(x) \neq \delta(x)\}$.*

If F is empty we say that (m, λ) is a linear involution without flips. If F is non-empty and differs from the set of double letters, we say that (m, λ) is a linear involution with flips.

Notice that the case where F is exactly the set of double letters in m (and is not empty) does not appear. Thus is due to the fact that when associating a linear involution to a measured foliation, one can exclude it [DN90]. This last condition is necessary in the proof to assert that there exists an open set of length parameter for which the involution has a flipped periodic point. It would be interesting to describe the generic dynamical behaviour of this last remaining case, but this is beyond the scope of this article.

Definition 3.4 (Interval exchange transformation)**.** *An interval exchange is a linear involution (m, λ) such that for all $x \in \mathcal{A}$, $|\nu|_x = |\omega|_x = 1$, i.e. all letters are simple. If moreover $\epsilon \equiv 1$ we say that (m, λ) is an interval exchange without flips, otherwise with flips.*

Definition 3.5. *Let $(\nu, \omega, \epsilon, \lambda)$ be a linear involution. We denote $\ell := \lambda(\nu) = \lambda(\omega)$ and the interval $I = [0, \ell[$. For each label $l \in \mathcal{A}$, we define two points ξ_l^0, ξ_l^1 in the interval I together with a number $\sigma \in \{0, 1\}$.*

- *If l occurs twice in ν at indices $1 \leq p < q \leq |\nu|$, then:*

$$\xi_l^0 = \sum_{i=1}^{p-1} \lambda_{\nu_i}, \quad \xi_l^1 = \sum_{i=1}^{q-1} \lambda_{\nu_i}, \quad \text{and} \quad \sigma_l = 0.$$

- *If l occurs twice in ω at indices $1 \leq p < q \leq |\omega|$, then:*

$$\xi_l^0 = \sum_{j=1}^{p-1} \lambda_{\omega_j}, \quad \xi_l^1 = \sum_{j=1}^{q-1} \lambda_{\omega_j}, \quad \text{and} \quad \sigma_l = 0.$$

- *If l occurs once in ν at index $1 \leq p \leq |\nu|$ and once in ω at index $1 \leq q \leq |\omega|$, then:*

$$\xi_l^0 = \sum_{i=1}^{p-1} \lambda_{\nu_i}, \quad \xi_l^1 = \sum_{j=1}^{q-1} \lambda_{\omega_j}, \quad \text{and} \quad \sigma_l = 1.$$

For $c \in \mathbb{Z}/2\mathbb{Z}$ and $l \in \mathcal{A}$, we denote by I_l^c the subinterval $]\xi_l^c, \xi_l^c + \lambda_l[\subset I$. Consider the unique linear map $\widetilde{f}_l^c : I_l^c \to I_l^{c+1}$ with constant derivative equal to $\epsilon(l)$. It can be explicitly expressed for all $x \in I_l^c$ by

$$\widetilde{f}_l^c(x) = \xi_l^{c+1} + \epsilon(l) \cdot \left(x - \xi_l^c - \frac{\lambda_l}{2} \right) + \frac{\lambda_l}{2}.$$

Let S be the set of all points ξ_l^c. We define an involution $\widetilde{f}$ on $I \setminus S \times \mathbb{Z}/2\mathbb{Z}$ by, for $x \in I_l^c$ and $\sigma_0 \in \mathbb{Z}/2\mathbb{Z}$,

$$\widetilde{f}(x, \sigma_0) = \left(\widetilde{f}_l^c(x), \sigma_0 + \sigma_l \right).$$

Remark 3.6. *This associated map motivates the name for the couple of signed matching and length vector. They correspond to linear involutions with and without flips, as considered in [DN90] and [BL09] respectively. In the case of interval exchange, the map always changes the element in $\mathbb{Z}/2\mathbb{Z}$ and can be factored into a translation map on the interval I, as defined in [Yoc10].*

Moreover, this association can clearly be performed in the other direction, from the involution map to a signed matching with a length vector.

The existence of a linear involution is central in the definition of Rauzy–Veech induction. But for some signed matching the condition on lengths can clearly not be met. For instance, see the matching with all double letters on top in Figure 3.

Definition 3.7. *We say a signed matching $m = (\nu, \omega, \epsilon)$ is balanced if m is an interval exchange or there exists a letter a in ν and b in ω such that $|\nu|_a = |\omega|_b = 2$.*

Proposition 3.8. *A signed matching m is balanced if and only if there exists a length parameter λ such that (m, λ) is a linear involution.*

Proof. Assume without loss of generality that all letter α in ω are such that $|\omega|_\alpha = 1$, then $\lambda(\nu) - \lambda(\omega) = 2\#\{\lambda_\beta \mid \beta \in \mathcal{A} \text{ and } |\nu|_\beta = 2\} = 0$. Which can only be true if the set of double letters is empty.

Conversely, the existence of a length vector is straightforward for interval exchanges and when there are double intervals on top and bottom. □

We denote by:

- $\Sigma(\mathcal{G}_n)$ the set of signed matching on n letters.
- $\Sigma^u(\mathcal{G}_n)$ the subset of unbalanced signed matching.
- $\Sigma^0(\mathcal{G}_n)$ the subset of signed matching such that either it is unbalanced or the last letters of the words ν and ω are equal.

For $x, y \in \mathcal{A}$, let us introduce the substitutions

$$s^1_{x,y} : x \to x \cdot y \quad \text{and} \quad s^{-1}_{x,y} : x \to y \cdot x.$$

For all non-empty word w we denote by $\widetilde{w}$ the same word to which we have removed the last letter.

Definition 3.9. *The Rauzy–Veech induction for $n \geq 2$ is the map*

$$\begin{aligned} \mathcal{R}_n : \qquad \Sigma(\mathcal{G}_n) \setminus \Sigma^0(\mathcal{G}_n) \times \mathbb{R}^n_+ &\longrightarrow \Sigma(\mathcal{G}_n) \times \mathbb{R}^n_+ \\ (\nu, \omega, \epsilon, \lambda) &\longmapsto (\nu', \omega', \epsilon', \lambda'). \end{aligned}$$

where for α, β the (distinct) last letters of ν and ω, the image is defined as follows :

- *If $\lambda_\alpha > \lambda_\beta$,*

$$\begin{aligned} \nu' &= s^{\epsilon(\alpha)}_{\alpha,\beta}(\widetilde{\nu}) \cdot \alpha, & \epsilon'(\beta) &= \epsilon(\alpha) \cdot \epsilon(\beta), \\ \omega' &= s^{\epsilon(\alpha)}_{\alpha,\beta}(\widetilde{\omega}), & \lambda'_\alpha &= \lambda_\alpha - \lambda_\beta. \end{aligned}$$

- *If $\lambda_\beta > \lambda_\alpha$,*

$$\begin{aligned} \nu' &= s^{\epsilon(\beta)}_{\beta,\alpha}(\widetilde{\nu}), & \epsilon'(\alpha) &= \epsilon(\beta) \cdot \epsilon(\alpha), \\ \omega' &= s^{\epsilon(\beta)}_{\beta,\alpha}(\widetilde{\omega}) \cdot \beta, & \lambda'_\beta &= \lambda_\beta - \lambda_\alpha. \end{aligned}$$

The coordinates that are not mentioned for ϵ' and λ' are kept unchanged.

Invariant linear form The difference $\lambda(\nu) - \lambda(\omega)$ is preserved by the Rauzy–Veech induction. Classically, the Rauzy–Veech induction is only defined in the case $\lambda(\nu) = \lambda(\omega)$ on maps associated to linear involution or interval exchanges.

In other terms, the family linear forms defined to each vertex (ν, ω, ϵ) of $\mathcal{G}_n$ defined for all $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ by

$$\phi_{(\nu,\omega,\epsilon)}(\lambda) := \lambda(\nu) - \lambda(\omega) \tag{2}$$

is preserved by composition with the induction. Hence its kernels form an invariant family of linear subspaces.

Proposition 3.10. *The kernel of the linear form intersects the positive cone if and only if m is balanced.*

Geometric interpretation Let us mention here that there is a geometric interpretation of these maps. It is not necessary to our definition but may help the reader to understand its intuition. As in Section 2.1 of [BL09], the linear involution can be seen as the first return map for a foliation on a surfaces on a transverse interval. The interval is duplicated to separate cases where the leaf arrives at the top or bottom of the interval.

Let s is the map switching values 0 and 1 in the second coordinate, *i.e.* for $(x, \sigma_0) \in I \setminus S \times \mathbb{Z}/2\mathbb{Z}$ by $s(x, \sigma_0) = (x, \sigma_0 + 1)$, the orbits of the composed map $s \circ \widetilde{f}$ correspond to the intersection of the leaves of the foliation with the interval. Such maps associated to a foliation depend on the choice of interval and Rauzy–Veech induction is a natural induction which builds up from a linear involution another one implied by the first return map of the same foliation on a different interval.

The following is proved in Section 2.2 of [BL09].

Proposition 3.11. *Let $L = (\nu, \omega, \epsilon, \lambda)$ be a linear involution. The linear involution L' is the image by Rauzy–Veech induction of L if and only if its associated map $s \circ \widetilde{f}_{L'}$ is the first return map of $s \circ \widetilde{f}_L$ on*

$$]0, \max(\lambda(\widetilde{\nu}), \lambda(\widetilde{\omega}))[\times \mathbb{Z}/2\mathbb{Z}.$$

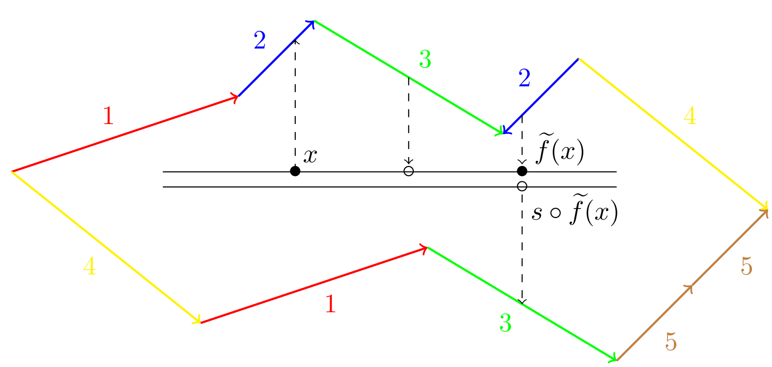


Figure 6: Linear involution as a first return map of a vertical foliation.

Definition 3.12. *We say L contains a Möbius strip if there exists a subinterval of $J \subset [0, \ell[$ and an integer n such that for all $x \in J$ and $\sigma \in \mathbb{Z}/2\mathbb{Z}$, we have $(s \circ \widetilde{f}_L)^n(x, \sigma) = (-x, \sigma)$.*

Geometrically this indeed corresponds to the embedding of a Möbius strip in the vertical foliation of Figure 6. Notice in particular, that this property is preserved by Rauzy induction. We will see in the following, using renormalization by the induction, that this property is generic for LI with flips.

Persistance of flips Let us indulge in a simple but key observation: to create a new flip in the matching, a non-flipped letter should lose against a flipped one. Conversely, to make a flip disappear, a flipped letter has to lose against another flipped letter.

This implies in particular that an interval exchange or linear involution with flip cannot be sent to one without flip. In the geometric interpretation, that can be seen from the fact that the underlying surface is preserved and there is flip if and only if the surface is non-orientable.

3.2 Rauzy–Veech as win-lose induction

Rauzy–Veech induction can be seen as the win-lose map associated to graph $\mathcal{G}_n$ whose vertices are all the signed matching in $\Sigma(\mathcal{G}_n)$. Edges going out of a vertex associated to a given signed matching $(\nu, \omega, \epsilon) \in \Sigma(\mathcal{G}_n) \setminus \Sigma^0(\mathcal{G}_n)$ are defined in Figure 7. The change on ϵ maps are not written down to simplify the presentation but are clear from the definition of the induction given above.

The image signed matching by Rauzy–Veech induction can be in $\Sigma^0(\mathcal{G}_n)$. In this case, the Rauzy–Veech induction is not defined and stops. Such vertices are thus defined in $\mathcal{G}_n$ to have no outgoing edges.

$$\begin{pmatrix} s^{\epsilon(\alpha)}_{\alpha,\beta}(\widetilde{\nu}) \cdot \alpha \\ s^{\epsilon(\alpha)}_{\alpha,\beta}(\widetilde{\omega}) \end{pmatrix} \xleftarrow{\quad\beta\quad} \begin{pmatrix} \nu \\ \omega \end{pmatrix} \xrightarrow{\quad\alpha\quad} \begin{pmatrix} s^{\epsilon(\beta)}_{\beta,\alpha}(\widetilde{\nu}) \\ s^{\epsilon(\beta)}_{\beta,\alpha}(\widetilde{\omega}) \cdot \beta \end{pmatrix}$$

Figure 7: Outgoing edges for a given signed matching

Property 3.13. *For a vertex $v \in \Sigma(\mathcal{G}_n)$, if there is a letter in the labels of E_v that is not in $\mathcal{L}$, this property is preserved along $\mathcal{L}$-factor paths in $\mathcal{G}_n$ or a subgraph of it.*

Proof. It is a key property of Rauzy–Veech induction that a letter winning along an edge from v to v' appears in the labels of edges going out of the ending vertex v'. But by the definition of $\mathcal{L}$-factor path of a subgraph, there is no letter in which a letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$. Thus, as there is at least one letter not in $\mathcal{L}$ going out of v, no matter whether the losing letter is in $\mathcal{L}$ or not, the winning letter is not. And this is true all along the branch path. □

Property 3.14. *Consider a path γ starting with an edge e from vertex v. In $\mathcal{G}_n$, if α wins against e, then either it is contained in $l(E_{\gamma \cdot v})$ or a letter in this latter set wins against α along γ.*

Proof. Again we use the fact that the winning label remains in the outgoing edge of the next vertex. Then notice that if β wins against α along γ_1 and δ wins against β along the next edge e' then δ wins against α along $\gamma_1 \cdot e'$ and δ appears in labels of the edges going out of the ending vertex of e'. We then prove the proposition by induction. □

3.2.1 Linear subspace restriction

As observed in Equation (2), the length difference between the top and bottom interval is preserved. This relation is computed by a linear form ϕ_v at vertex $v = (\nu, \omega, \epsilon)$. For interval exchanges, this form is zero, but for linear involutions, it induces an induction on the subgraph of vertices such that $\mathbb{R}_+^{\mathcal{A}} \cap \ker \phi_v$ has non-empty interior as explained in Section 2.1.1. By Proposition 3.8, the corresponding set of vertices are balanced matching. Using this proposition, one can also characterize vertices with two outgoing edges.

Proposition 3.15. *A signed matching (ν, ω, ϵ) has two outgoing edges in G^ϕ if and only it none of the ending labels of ν and ω is the unique double letter in the word. We say such vertices are non-constrained.*

Let $\mathcal{G}_n^\phi$ be the subgraph of $\mathcal{G}_n$ from which we remove vertices in $\Sigma^u(\mathcal{G}_n)$ and edges pointing to them. We denote them respectively by G^ϕ and G.

Projective measures In the remaining of this subsection, we check a key inequality on mesures induced by Lebesgue measure on polytopes Δ_v^ϕ. They are central to apply Theorem A and Theorem A' to hyperplane restrictions (see Remarks 1.10 in [Fou24a]).

Let us denote by $\mathcal{S}$ the set of simple letters and $\mathcal{D}_T$ and $\mathcal{D}_B$ the set of double letters in ν and ω respectively. We thus have a splitting $\mathcal{A} = \mathcal{S} \sqcup \mathcal{D}_T \sqcup \mathcal{D}_B$. Polytopes Δ_v^ϕ are the convex hull of $\mathbb{R}_+$-rays generated by the vectors

- b_α with $\alpha \in \mathcal{S}$
- $b_\alpha + b_\beta$ with $\alpha \in \mathcal{D}_T$ and $\beta \in \mathcal{D}_B$.

Where b_α with $\alpha \in \mathcal{A}$ is the canonical basis of $\mathbb{R}^{\mathcal{A}}$.

Definition 3.16. *Let $q \in \mathbb{R}_+^{\mathcal{A}}$, and let ν_q be the Borel measure on Δ, defined as follows: for any subset $A \subset \Delta$,*

$$\nu_q(A) := \mathrm{Leb}(p^{-1}A \cap \Lambda_q)$$

where $p : \mathbb{R}_+^{\mathcal{A}} \to \mathbb{R}_+^{\mathcal{A}}/\sim = \Delta$ is the quotient map and $\Lambda_q = \left\{ v \in \mathbb{R}_+^{\mathcal{A}} \,\middle|\, \langle q, v \rangle < 1 \right\}$.

Consider Leb^{ϕ_v} the Lebesgue measure on $\ker \phi$ normalized by the linear form, using a common Lebesgue measure on $\mathbb{R}$. One can again define for $q \in \mathbb{R}_+^{\mathcal{A}}$, $\nu_q^{\phi_v}$ the Borel measure on Δ_v^ϕ, for any subset $A \subset \Delta_v^\phi$, by $\nu_q^{\phi_v}(A) := \mathrm{Leb}^{\phi_v}(p^{-1}A \cap \Lambda_q)$. And the probability measure on paths in G^ϕ by $\mathbb{P}_q^{\phi_v}(\gamma) = \dfrac{\nu_q^{\phi_v}\left(\Delta_{\phi_v}^\gamma\right)}{\nu_q^{\phi_v}\left(\Delta_{\phi_v}\right)}$.

Consider for $\alpha \in \mathcal{D}_T, \beta \in \mathcal{D}_B$ scalars such that $\lambda_{\alpha,\beta}$ such that

$$\sum_{\alpha \in \mathcal{D}_T,\, \beta \in \mathcal{D}_B} \lambda_{\alpha,\beta} = 1.$$

Notice that

$$\sum_{\alpha \in \mathcal{D}_T,\, \beta \in \mathcal{D}_B} \lambda_{\alpha,\beta}(b_\alpha + b_\beta) = \sum_{\alpha \in \mathcal{D}_T} \mu_\alpha b_\alpha + \sum_{\beta \in \mathcal{D}_B} \nu_\beta b_\beta.$$

Where $\mu_\alpha = \sum_{\beta \in \mathcal{D}_B}$ and $\mu_\beta = \sum_{\alpha \in \mathcal{D}_T}$. As $\sum_{\alpha \in \mathcal{D}_T} \mu_\alpha = 1$ and $\sum_{\beta \in \mathcal{D}_B} \nu_\beta = 1$ these vectors define an element of Δ_n and Δ_m. Which leads to the following proposition.

Proposition 3.17. *The convex hull of vectors $(b_\alpha + b_\beta)_{\alpha \in \mathcal{D}_T, \beta \in \mathcal{D}_B}$ is isomorphic to the product of two simplices $\Delta_n \times \Delta_m$ composed of respectively $n = |\mathcal{D}_T|$ and $m = |\mathcal{D}_B|$ vertices.*

There is a clever construction of a triangulation for product of simplices, called the staircase triangulation. Details can be found in Theorem 6.2.13 of [DLRS10]. We will only use the following here.

Theorem 3.18. *Assume Δ_n and Δ_m are the convex hull respectively of vertices $a_1, \dots, a_n$ and $b_1, \dots, b_m$. There exists a triangulation of $\Delta_n \times \Delta_m$ such that all simplices of the triangulation contain a_1 and b_1.*

This enables us to prove the following.

Proposition 3.19. *For a non-constrained vertex v, consider α, β the rightmost labels of its top and bottom words and the half space $D_{\alpha,\beta} = \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha < \lambda_\beta\}$. For all $K > 0$, there exists $\sigma > 0$ such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$ satisfying $q_\delta \leq K q_\alpha$ for all $\delta \in \mathcal{A}$,*

$$\frac{1}{2K+1} \leq \frac{\nu_q(\Delta_v^\phi \cap D_{\alpha,\beta})}{\nu_q(\Delta_v^\phi)} \leq \left(\frac{q_\alpha}{q_\alpha + q_\beta} \right)^\sigma.$$

Proof. By the previous theorem, there exists a triangulation of Δ_v^ϕ by simplices which all contain the following two vertices :

- if α and β are simple, b_α and b_β;
- if α is simple and β is double, b_α and $b_\beta + b_\delta$ for some other double label δ;
- if β is simple and α is double, $b_\alpha + b_\delta$ and b_β for some other double label δ;
- if α and β are double, $b_\alpha + b_\delta$ and $b_\beta + b_{\delta'}$ for some other double labels δ, δ'.

Notice that the existence of other double labels is implied by the vertex being non-constrained. Let us prove the inequality in the last case, the others being similar.

Let us denote by Δ one simplex of this triangulation and $b_\alpha + b_\delta, b_\beta + b_{\delta'}, e_3, \dots, e_d$ its vertices. The intersection of Δ with $D_{\alpha,\beta}$ is contained in the simplex Δ_α defined as the convex hull of $b_\alpha + b_\beta + b_\delta + b_{\delta'}, b_\beta + b_{\delta'}, e_3, \dots, e_d$.

Notice that

$$\frac{\nu_q(\Delta \cap D_{\alpha,\beta})}{\nu_q(\Delta)} \leq \frac{\nu_q(\Delta_\alpha)}{\nu_q(\Delta)} = \frac{q_\alpha + q_\delta}{q_\alpha + q_\beta + q_\delta + q_{\delta'}} \leq \frac{q_\alpha + q_\delta}{q_\alpha + q_\beta + q_\delta}.$$

By assumption $q_\delta, q_\beta \leq K q_\alpha$, thus there exists $\sigma > 0$ (depending only on K) such that

$$\frac{\nu_q(\Delta \cap D_{\alpha,\beta})}{\nu_q(\Delta)} \leq \left(\frac{q_\alpha}{q_\alpha + q_\beta} \right)^\sigma.$$

The proof of this statement can be found in details in Proposition 2.20 of [Fou24a]. On the other hand, intersecting with the complementary half-plane, we have

$$\frac{\nu_q(\Delta \cap D_{\beta,\alpha})}{\nu_q(\Delta)} \leq \frac{q_\beta + q_{\delta'}}{q_\beta + q_{\delta'} + q_\alpha} \leq \frac{2K}{2K+1} \quad \text{thus} \quad \frac{\nu_q(\Delta \cap D_{\alpha,\beta})}{\nu_q(\Delta)} \geq \frac{1}{2K+1}.$$

By splitting the measure on the simplices of its triangulation, we see that the ratio $\dfrac{\nu_q(\Delta_v^\phi \cap D_{\alpha,\beta})}{\nu_q(\Delta_v^\phi)}$ is a weighted average of the ratio for these simplices. Which concludes the proof. □

3.2.2 Subgraph restriction

A strongly connected component of a directed graph is by definition a maximal subgraph such that all distinct vertices v and v' have a directed path from v to v' and from v' to v.

Definition 3.20. *An* irreducible component *is a strongly connected components of $\mathcal{G}_n^\phi$ for which the set of labels of outgoing edges in G is the whole alphabet $\mathcal{A}$. A signed matching in such a component is called* irreducible.

Remark 3.21. *If a strongly connected component is not irreducible, one can see the subgraph as an irreducible component for LI on the labels that appear. The other parameters remain unchanged by the induction.*

> In the following we fix such strongly connected component and denote by F the corresponding subgraph of G^ϕ.

Remark 3.22. *For interval exchanges, irreducible signed matching (ν, ω, ϵ) is fully characterized by the fact that their is no non-trivial decomposition $\nu = \nu^1 \cdot \nu^2$ and $\omega = \omega^1 \cdot \omega^2$ such that the set of labels appearing in ν^1 and ω^1 (and thus in ν^2 and ω^2) are equal (see e.g. [Yoc10]).*

For linear involutions without flips, irreducible signed matching are characterized in [BL09] Definition 3.1. Indeed, a dynamically irreducible generalized permutation with admissible lengths must become strongly irreducible in finite time. Thus, if the permutation is not strongly irreducible, there is no loop based at this permutation along which all labels appear. Since it would induce a positive matrix and an admissible length vector which never becomes strongly irreducible. In particular, it does not belong to an irreducible component.

It is also proved in that work that such objects come from a geometric model (as the first return map on an interval of the vertical foliation of a half-translation surfaces).

It would be interesting to address the following question.

Question 3.23. *Characterize irreducible signed matchings with flips.*

Nonetheless, it is not necessary to the present work to understand this key combinatorial link between the components of the diagram and properties of the signed matchings. We only need here the following combinatorial characterisation of the subgraph.

Proposition 3.24. *Irreducible components are non-degenerating.*

As a warm up, let us prove it for full graphs, *i.e.* for the classical Rauzy diagram on IETs and its generalization to LI without flips.

Consider a subset $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and vertex v in a strongly connected component $\mathscr{C}$ of $G_{\mathcal{L}}^{\perp}$.

- If both outgoing edges of v are labeled in $\mathcal{L}$, by irreducibility of the signed matching, there exists a path to a vertex with at least one outgoing edge not labeled in $\mathcal{L}$. Consider the shortest factor pointing to such a vertex, it must be labeled in $\mathcal{L}$ and leave $\mathscr{C}$ by Property 3.13. Thus v satisfies Condition 1.
- If at least one of the outgoing edges of v is not labeled in $\mathcal{L}$, the vertex obviously satisfies Condition 2.

In the general case, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, we define a subset of vertices of $F_{\mathcal{L}}$.

$$\widetilde{V}_{\mathcal{L}} = \left\{ v \in F_{\mathcal{L}} \mid E_v^G \cap \overline{\mathcal{L}} = \emptyset \text{ or } E_v^F \cap \mathcal{L} = \emptyset \text{ or } |E_v^F| = 2 \right\}.$$

We start by showing that this constitute a filling factoring family.

Proposition 3.25. *A $\mathcal{L}$-factor path associated to $\{\widetilde{V}_{\mathcal{L}}\}$ cannot loop.*

Proof. Assume there is a loop that does not contain any vertex in $\widetilde{V}_{\mathcal{L}}$. Then it is composed of non branching vertices in F and their unique outgoing edge is labeled in $\overline{\mathcal{L}}$. The loop then composes the whole strongly connected component and is labeled in a strict subset of the alphabet. Which contradicts the irreducibility property. □

Proof of Proposition 3.24. Let γ be a $\mathcal{L}$-factor path starting at a vertex v and contained in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$.

If $E_v^G \cap \overline{\mathcal{L}} = \emptyset$. By irreducibility, there exists a path γ starting at v in F which contains ends at a vertex with a least one outgoing label in G which is not in $\mathcal{L}$. Up to taking a prefix of γ, we assume its end vertex is the first vertex to satisfy this condition. Thus all previous branching vertices have both their outgoing edges labeled in $\mathcal{L}$. By Property 3.13 the path γ must leave the strongly connected component $\mathscr{C}$ of v. Condition 1 is then satisfied.

If $E_v^F \cap \mathcal{L} = \emptyset$. Let $\alpha \in \overline{\mathcal{L}}$ be the first label of γ and β the label of the other edge in G. Then β must also be in $\overline{\mathcal{L}}$ since otherwise a letter in $\mathcal{L}$ would win against a letter in $\overline{\mathcal{L}}$ along γ. Again by Property 3.13, in a $\mathcal{L}$-factor path starting at v, all vertices have at least one outgoing edge in G not labeled in $\mathcal{L}$. If the path is composed of only one edge, this falls in Condition 2.

Otherwise, in intermediate steps there is always one edge labeled in $\mathcal{L}$ in F which must lose. And the winning letter along each edge of the path is always β since is must be preserved by definition of Rauzy–Veech induction. Thus Condition 4 is satisfied.

If $|E_v^F| = 2$. Assuming the other conditions for the set of vertices are false, v has two outgoing edges respectively labeled by $\alpha \in \mathcal{L}$ and $\beta \in \overline{\mathcal{L}}$. Again, along every edges of γ, the winning letter must be β. Moreover the losing letter has to be in $\overline{\mathcal{L}}$ for the vertex no to be in $\widetilde{V}_{\mathcal{L}}$. The path thus satisfies Condition 3. □

Remark 3.26. *Lemma 3.6 in [Fou24a] implies that there exists a positive path in an irreducible component. In other terms, there is a path in the component along which all letters in $\mathcal{A}$ lose. Thus a component is irreducible if and only if all letters in $\mathcal{A}$ appear as labels of its edges inside the connected component.*

3.3 Strongly connected components

Definition 3.27. *We say a strongly connected component of F is extremal if no edge in G^ϕ leaves it. We call it trivial if it is composed of a unique vertex with no outgoing edges.*

Remark. *The name comes from the fact that the component is actually extremal in the condensation graph of strongly connected component.*

Proposition 3.28 ([BL09])**.** *An irreducible component for LI without flips is extremal.*

Proof. The induction in such component admits a natural extension with zippered rectangle (as explained in the next section). In this setting the natural extension preserves Lebesgue measure which is known to be finite and non-zero since Veech [Vee90]. Thus, the induction must be recurrent and no edge can go out of the given strongly connected component. □

In the case without flips, by Remark 3.22, all strongly connected components are extremal. This concept will only be useful in the study of flipped cases.

Proposition 3.29. *An irreducible component for a LI with flips is not extremal.*

Proof. Assume for contradiction that all edges in $\mathcal{G}_n^\phi$ remain in the component. Consider a signed matching in it. By assumption, there exists a flipped interval labeled by α in the matching.

If α is simple, follow a path until α appears in the label of outgoing edges of the ending vertex — it exists by irreducibility. At this vertex α is still simple in the corresponding signed matching. It should thus be allowed for α to win in $\mathcal{G}_n^\phi$ since it does not change the number of double labels in each word. But after a finite number of steps where α wins the other ending label will be α since the losing letter goes to the left of the twin α interval. Which leads to a contradiction.

Assume now that there are no flipped simple interval. Consider α double and flipped. By definition of linear involution with flips, there exists another double interval β which is not flipped. Recalling Proposition 3.8, the signed matching must be balanced, so we can assume α and β are in two different words. As before, follow a path until α or β appear in the labels of outgoing edges of the ending vertex. Assume this letter is α (the other case is similar), follow a path which makes this letter win until the other appears as the ending letter. Such path should be in $\mathcal{G}_n^\phi$ since to be constrained, the other ending label should be double and the only double letter in the word, thus this could only happen if the other ending letter is β. In this configuration, if α and β are both the unique double label in their word, the length condition implies equality of their length and the vertex has no outgoing vertex. Otherwise, one of them could win, making the other label simple and flipped. Both cases lead again to a contradiction. □

Proposition 3.30. *Consider $L = (\nu, \omega, \epsilon, \lambda)$ a LI with flips in a non-trivial extremal component. Let us decompose $\mathcal{A}_l \sqcup \mathcal{A}_r = \mathcal{A}$ by respectively the subset of letters appearing as labels of outgoing edges in the component and its complementary set.*

There exists a decomposition of the words

$$\nu = \nu_{\mathcal{A}_r} \cdot \nu_{\mathcal{A}_l} \quad \text{and} \quad \omega = \omega_{\mathcal{A}_r} \cdot \omega_{\mathcal{A}_l}$$

of non-trivial factors with letters respectively in $\mathcal{A}_r$ and $\mathcal{A}_l$ such that the component is isomorphic to the diagram generated by $(\nu_{\mathcal{A}_l}, \omega_{\mathcal{A}_l}, \epsilon_{\mathcal{A}_l})$ where $\epsilon_{\mathcal{A}_l}$ is the restriction of ϵ to $\mathcal{A}_l$.

Moreover, the $s \circ \widetilde{f}_L$ is conjugated to the union of the maps $s \circ \widetilde{f}_{L_l}$ and $s \circ \widetilde{f}_{L_r}$. Where

$$L_l = (\nu_{\mathcal{A}_r}, \omega_{\mathcal{A}_r}, \epsilon_{\mathcal{A}_r}, \lambda_{\mathcal{A}_r}) \quad \text{and} \quad L_r = (\nu_{\mathcal{A}_l}, \omega_{\mathcal{A}_l}, \epsilon_{\mathcal{A}_l}, \lambda_{\mathcal{A}_l}).$$

Notice that L_l is again a LI with flips whereas L_r is a LI without flips.

Proposition 3.31. *Let $L = (\nu, \omega, \epsilon, \lambda)$ be a LI in a trivial extremal component, and let α, β be its ending letters.*

If $\alpha = \beta$, the map $s \circ \widetilde{f}_L$ is conjugated to the union of $\epsilon(\alpha) \cdot id$ on $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ and the map $s \circ \widetilde{f}_{L'}$ with

$$L' = (\widetilde{\nu}, \widetilde{\omega}, \epsilon_{\overline{\alpha}}, \lambda_{\overline{\alpha}})$$

where $\overline{\alpha}$ stand for the restriction to the complement of α in $\mathcal{A}$.

If $\alpha \neq \beta$, $\lambda(\alpha) = \lambda(\beta)$ and the first return map of $s \circ \widetilde{f}_L$ to the subinterval $[0, \ell - \lambda(\alpha)[$ is conjugated to the $s \circ \widetilde{f}_{L'}$ with

$$L' = (s^0_{\alpha,\beta}(\widetilde{\nu}), s^0_{\alpha,\beta}(\widetilde{\omega}), \epsilon_{\overline{\alpha}}, \lambda_{\overline{\alpha}})$$

where $s^0_{\alpha,\beta}$ is the substitution which transforms letter β into α.

If $\alpha = \beta$ and $\epsilon(\alpha) = -1$ the last interval, corresponding to α, is a Möbius strip. Thus in this case we have shown that the LI we started doing the induction with contains a Möbius strip.

In other cases, the last interval is either a flat cylinder or a transitive part for the dynamics. But in both cases, L contains a Möbius strip if and only if L' does and L' is again a LI with flips.

Remark 3.32. *This decomposition discussion is missing in [DN90] but necessary for a complete proof.*

As a consequence, of Proposition 3.29, Theorem B and B' we can bound the Hausdorff dimension of IET and LI with flips which do not contain a Möbius strip.

Theorem 3.33. *Let m be the signed matching of an IET (resp. LI) with flips in an irreducible component. The set of lengths $\lambda \in \mathbb{R}^{\mathcal{A}}_+$ such that a (m, λ) does not contain a Möbius strip is strictly smaller than $|\mathcal{A}|$ (resp. $|\mathcal{A}| - 1$).*

Proof. The non-degenerating property of the graph implies by the work in [Fou24a] that the Hausdorff dimension of length parameter remaining in a given irreducible component of the Rauzy diagram for IET or LI with flips is strictly smaller than the dimension of its ambient space.

If there exists edges leaving the component, one can decompose the parameter space into this set of parameters remaining in the irreducible component and a countable union of polytopes corresponding to finite paths leaving the component.

The new component corresponding to such a polytope can either be extremal — with no edges leaving it — or not. If it is not, consider the set of labels appearing in outgoing edges of its vertices. By Remark 3.21, one can apply the result to the corresponding subspace of parameters. By stability of the Hausdorff dimension under countable union, this shows that the space of parameters contains a union of polytopes which go in finite time to an extremal component and which complementary set is of lower Hausdorff dimension.

For polytopes in an extremal component, the LI either contains a Möbius strip or we consider its other part which does not interact with Rauzy induction — denoted L_l or L' above. They are LI with flips with strictly less interval. One then concludes by induction. □

4 Natural extensions

4.1 Zippered rectangles

In the case of interval exchange transformation a geometrical parametrization of the natural extension of Rauzy–Veech induction has been introduced by Veech in

[Vee78]. He has named this construction zippered rectangles, expressing in a visual way the intuition behind it. This construction was generalized to linear involution in [BL09].

Assume ν, ω and ϵ define a signed matching associated to an interval exchange of linear involution (hence ϵ is fully determined by the words). Assume there is a (half-)translation surface that suspends the associated map or in other words that one can find n vectors ν in $\mathbb{C}$ labeled in $\mathcal{A}$ such that

- for all $\alpha \in \mathcal{A}$, $\mathrm{Re}(\nu_\alpha) > 0$,
- for all $1 < k < |\nu|$, $\displaystyle\sum_{\alpha \in \nu(1\ldots k)} \mathrm{Im}(\nu_\alpha) > 0$,
- for all $1 < k < |\omega|$, $\displaystyle\sum_{\alpha \in \omega(1\ldots k)} \mathrm{Im}(\omega_\alpha) < 0$.

Where $w(1\ldots k)$ denotes the length k prefix of the word w.

Let us now consider the polygon obtained by representing these vectors starting at point 0 one after the other in both orderings given by ν and ω. We identify pairs of vectors with matching labels by translation or translation composed with central symmetry (when ϵ is respectively positive or negative). Then the obtained translation surface suspends the linear involution defined by the given signed matching and the lengths $(\mathrm{Re}(v_\alpha))_{\alpha \in \mathcal{A}}$.

On this surface, one can represent the suspension data by considering the first return of the vertical flow on the horizontal interval. Figure 8 represents these two constructions.

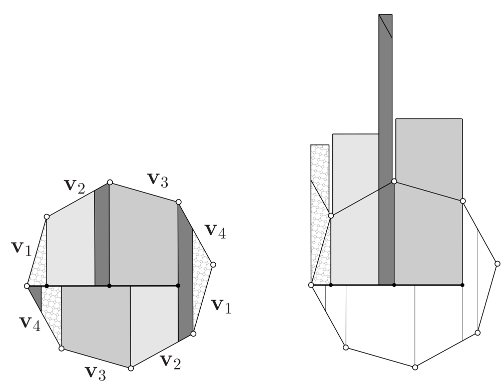


Figure 8: Zippered rectangle construction.

Let $\mathcal{A}_m$ be a marked alphabet containing two copies x_0, x_1 of each letter $x \in \mathcal{A}$. We assume moreover, up to choosing a marking, that ν and ω are words in $\mathcal{A}_m$ such that each copy of a letter appears exactly once in the concatenation of the words.

Let σ be the involution of $\{0,1\}$ which switches the two elements. For every $x \in \mathcal{A}$, let f_x and g_x be the bijections on $\{0,1\}$ defined as

$$f_x = \begin{cases} id & \text{if } \epsilon(x) = 1 \\ \sigma & \text{otherwise} \end{cases} \quad \text{and} \quad g_x = \begin{cases} id & \text{if } \delta(x) = 1 \\ \sigma & \text{otherwise} \end{cases}.$$

We introduce the notation $\iota_x(ij) := f_x(i)g_x(j)$.

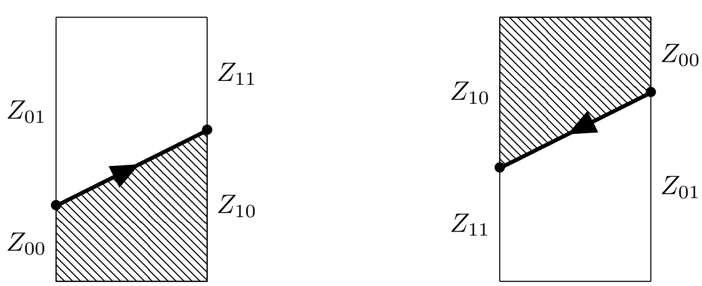
For each interval, we consider the rectangle formed by the suspension of the interval above the top and below the bottom domains corresponding to ν and ω. The corresponding side in the polygonal surface representation cuts it horizontally in a top and bottom half. Each vertical side are cut into two pieces — except for the rectangles meeting the leftmost singularity. We denote their length by Z_{ij} where i, j are 0 or 1 for respectively left, right and down, up interval.

In the wording of Veech, these numbers define *zip heights* which express the position of the singularities and enable us to reconstruct the surface. These heights satisfy the properties in the following definition.

Definition 4.1 (Zip functions)**.** *Let m be a signed matching of linear involution. Let $\alpha_k, \beta_l \in \mathcal{A}$ be the last letters of ν and ω respectively and γ_m, η_n the first letters. Zip functions for m are functions $Z_{00}, Z_{01}, Z_{10}, Z_{11} : \mathcal{A}_m \to \mathbb{R}$ such that*

1. $Z_{00} + Z_{01} = Z_{10} + Z_{11}$,
2. $Z_{ij}(x_{\sigma(k)}) = Z_{\iota_x(ij)}(x_k)$ *for all* $x_k \in \mathcal{A}_m$ *and* $i, j \in \{0, 1\}$,
3. $Z_{1i}(x_k) = Z_{0i}(y_l)$ *for all* $x_k, y_l \in \mathcal{A}_m$ *such that* $x_k y_l$ *is a factor of* w_i,
4. $Z_{10}(\alpha_k) = -Z_{11}(\beta_l)$ *and* $Z_{00}(\gamma_m) = Z_{01}(\eta_n) = 0$.
5. $Z_{10}(\alpha_k) + Z_{11}(\alpha_k) > 0$ *and* $Z_{00}(\beta_l) + Z_{01}(\beta_l) > 0$.
6. *for all* $x_k \in \alpha_m$ *and* $i, j \in \{0, 1\}$ *such that* $Z_{ij}(x_k)$ *or its twin from 2 does not appear in 4,*
$$Z_{ij}(x_k) > 0.$$

Condition 2 can be understood according to Figure 10.



(a) A (b) $-A$

Figure 9: Identification of the zip functions when applying -*id*.

We denote by $\mathcal{Z}_m$ the space of maps satisfying these conditions for a signed matching m. Notice that it is a subcone (stable by scalar multiplication by $\mathbb{R}_+$) of $(\mathbb{R}^{\mathcal{A}_m})^4$.

These functions induce a height map $h : \mathcal{A} \to \mathbb{R}_+$ defined for all $\alpha \in \mathcal{A}$ and $i, j \in \{0, 1\}$,
$$h(\alpha) := Z_{i0}(\alpha_j) + Z_{i1}(\alpha_j).$$
According to properties 1 and 2 of the definition, this does not depend on i, j.

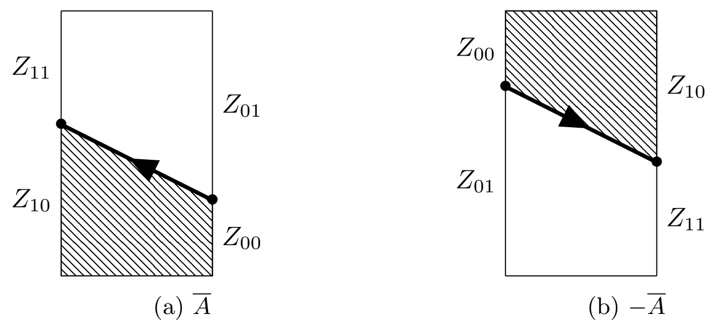


Figure 10: Identification when applying $z \mapsto \overline{z}$ and $z \mapsto -\overline{z}$.

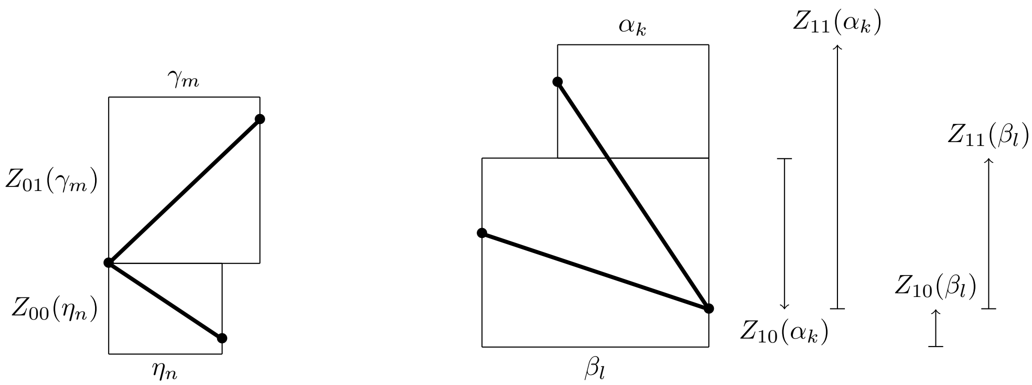


Figure 11: Zip functions around the leftmost and rightmost singularities.

Let

$$\mathcal{S}_n := \bigsqcup_{m \in \Sigma(\mathcal{G}_n) \setminus \Sigma^0(\mathcal{G}_n)} (\{m\} \times \mathbb{R}_+^n \times \mathcal{Z}_m)$$

be an extension of the parameter space $\Sigma(\mathcal{G}_n) \setminus \Sigma^0(\mathcal{G}_n) \times \mathbb{R}_+^n$ on which Rauzy–Veech induction is defined. One can extend Rauzy–Veech induction

$$\widehat{\mathcal{R}}_n : (\nu, \omega, \epsilon, \lambda, Z) \in \mathcal{S}_n \longmapsto (\nu', \omega', \epsilon', \lambda', Z') \in \mathcal{S}_n$$

Where $(\nu', \omega', \epsilon')$ and λ' are like in Definition 3.9 and Z' is defined by the formulas below.

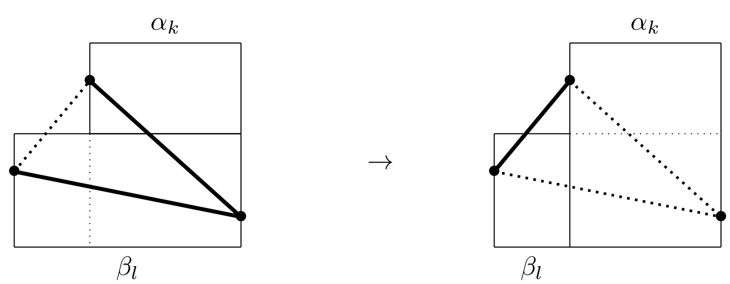
These formulas are motivated by an induction on the underlying surface of a zippered rectangles construction. We cut off a triangle on the left-hand side of the surface and glue one side of the triangle to the polygon.

Remark 4.2. *We express Rauzy–Veech induction on only one marking, the action on the other marking of the same letter is defined to preserve Property 2 in the definition. Other values that do not appear in the definition are unchanged.*

4.1.1 Extension of the induction.

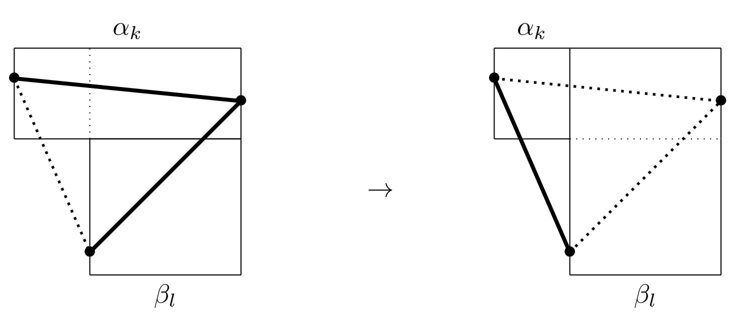
If bottom is winning,

$$
\begin{aligned}
Z'_{10}(\beta_l) &= Z_{00}(\alpha_k) && + h(\beta) \\
Z'_{11}(\beta_l) &= -Z_{00}(\alpha_k) \\
Z'_{\iota_\beta(00)}(\alpha_k) &= Z_{00}(\alpha_k) && + h(\beta) \\
Z'_{\iota_\beta(01)}(\alpha_k) &= Z_{01}(\alpha_k) \\
Z'_{\iota_\beta(10)}(\alpha_k) &= Z_{10}(\alpha_k) && + h(\beta) && = Z_{10}(\beta_l) \\
Z'_{\iota_\beta(11)}(\alpha_k) &= Z_{11}(\alpha_k) && && = Z_{11}(\beta_l) + h(\alpha)
\end{aligned}
$$



If top is winning,

$$
\begin{aligned}
Z'_{10}(\alpha_k) &= -Z_{01}(\beta_l) \\
Z'_{11}(\alpha_k) &= Z_{01}(\beta_l) && + h(\alpha) \\
Z'_{\iota_\alpha(00)}(\beta_l) &= Z_{00}(\beta_l) \\
Z'_{\iota_\alpha(01)}(\beta_l) &= Z_{01}(\beta_l) && + h(\alpha) \\
Z'_{\iota_\alpha(10)}(\beta_l) &= Z_{10}(\beta_l) && && = Z_{10}(\alpha_k) + h(\beta) \\
Z'_{\iota_\alpha(11)}(\beta_l) &= Z_{11}(\beta_l) && + h(\alpha) && = Z_{11}(\alpha_k)
\end{aligned}
$$



Let $e : m \to m'$ be an edge labeled by α. The formula above defines in particular a linear map H_e from $\left(\mathbb{R}^{\mathcal{A}_m}\right)^4$ to $\left(\mathbb{R}^{\mathcal{A}_{m'}}\right)^4$ and a vector v_e such that

$$Z' = H_e Z + h(\alpha) v_e =: \Theta_e(Z). \tag{3}$$

Where Θ_e is an affine map in Z.

4.1.2 Invertibility

We show that the affine map Θ_e is actually a bijection on its image, which is a simple subset of $\mathcal{Z}_{m'}$.

Definition 4.3. *Let*

$$\mathcal{Z}_m^0 := \{Z \in \mathcal{Z}_m \mid Z_{10}(\alpha_k) > 0\} = \{Z \in \mathcal{Z}_m \mid Z_{11}(\beta_l) < 0\}$$

and

$$\mathcal{Z}_m^1 := \{Z \in \mathcal{Z}_m \mid Z_{10}(\alpha_k) < 0\} = \{Z \in \mathcal{Z}_m \mid Z_{11}(\beta_l) > 0\}$$

where α_k, β_l denote the last letters of the words of m in the marked alphabet.

Proposition 4.4. *If the losing letter is on top (resp. bottom) the map Θ_e is a bijection from $\mathcal{Z}_m$ to $\mathcal{Z}_{m'}^0$ (resp. $\mathcal{Z}_{m'}^1$).*

Proof. This fact is clear when considering the geometric interpretation of the induction. But we will check it directly on the formulas to convince ourselves that they correspond to the construction.

Let us consider the case where bottom — letter β_l — is winning. The other case is similar. We prove $\Theta_e(\mathcal{Z}_m) \subset \mathcal{Z}_{m'}^0$.

Start by checking properties of Definition 4.1:

1. Property 1 in the definition is obviously preserved.
2. Property 2 is preserved since we defined the induction on the other marking using this property.
3. Two consecutive letters $x_r y_s$ remain consecutive in ν' and ω' if they are distinct from $\beta_{\sigma(l)}$ and α_k.

   If $\epsilon(\beta) = 1$ and $\delta(\beta) = 1$, a factor $\beta_{\sigma(l)} y_s$ appears in ν and we have two possibly new consecutive couple of letters $\beta_{\sigma(l)}\alpha_k$ and $\alpha_k y_s$ after induction. Remark from the equations that $Z'_{10}(\beta_l) = Z'_{\iota_\beta(00)}(\alpha_k) = Z'_{00}(\alpha_k)$ hence $Z'_{10}(\beta_{\sigma(l)}) = Z'_{10}(\beta_l) = Z'_{00}(\alpha_k)$. And $Z'_{10}(\alpha_k) = Z'_{\iota_\beta(10)}(\alpha_k) = Z_{10}(\beta_l) = Z'_{00}(y_s)$.

   If $\epsilon(\beta) = 1$ and $\delta(\beta) = -1$, there is a factor $\beta_{\sigma(l)} y_s$ in ω and we have two possibly new consecutive couple of letters $\beta_{\sigma(l)}\alpha_k$ and $\alpha_k y_s$ after induction. Again $Z'_{11}(\beta_{\sigma(l)}) = Z'_{\iota_\beta(00)}(\alpha_k) = Z'_{01}(\alpha_k)$ and $Z'_{11}(\alpha_k) = Z_{\iota_\beta(11)}(\beta_{\sigma(l)}) = Z'_{01}(y_s)$.

   If $\epsilon(\beta) = -1$ and $\delta(\beta) = -1$, there is a factor $x_r\beta_{\sigma(l)}$ in ω and we have at most two possibly new consecutive couple of letters $x_r\alpha_k$ and $\alpha_k\beta_{\sigma(l)}$ after induction. Then $Z'_{10}(\beta_l) = Z'_{\iota_\beta(00)}(\alpha_k)$ hence $Z'_{11}(\alpha_k) = Z'_{01}(\beta_{\sigma(l)})$. And $Z'_{11}(x_r) = Z_{01}(\beta_{\sigma(l)}) = Z_{10}(\beta_l) = Z'_{01}(\alpha_k)$.

If $\epsilon(\beta) = -1$ and $\delta(\beta) = 1$, there is a factor $x_r\beta_{\sigma(l)}$ in ν and we have at most two possibly new consecutive couple of letters $x_r\alpha_k$ and $\alpha_k\beta_{\sigma(l)}$ after induction. Again $Z'_{00}(\beta_{\sigma(l)}) = Z'_{\iota_\beta(00)}(\alpha_k) = Z'_{10}(\alpha_k)$ and $Z'_{10}(x_r) = Z_{00}(\beta_{\sigma(l)}) = Z_{10}(\beta_l) = Z'_{\iota_\beta(10)}(\alpha_k) = Z'_{00}(\alpha_k)$.

4. Let $\alpha'_{k'}$ be the last letter of ν' and $\eta'_{n'}$ the first letters of ω'. If $\alpha' \neq \alpha$ then $\alpha'_{k'}\alpha_k$ is a factor of ν and

$$Z'_{10}(\alpha'_{k'}) = Z_{10}(\alpha'_{k'}) = Z_{00}(\alpha_k) = -Z'_{11}(\beta_l).$$

Moreover,

$$Z'_{11}(\alpha'_{k'}) = Z_{11}(\alpha'_{k'}) = Z_{01}(\alpha_k)$$

thus

$$h'(\alpha') = Z'_{10}(\alpha'_{k'}) + Z'_{11}(\alpha'_{k'}) = Z_{00}(\alpha_k) + Z_{01}(\alpha_k) = h(\alpha) > 0.$$

If $\epsilon(\beta) = 1$ the first letters of ν and ω remain unchanged. If $\alpha' = \alpha$ then $\beta_{\sigma(l)}\alpha_k$ is a factor of ν' and

$$Z'_{1g_\beta(j)}(\alpha'_{k'}) = Z'_{\iota_\beta(1j)}(\alpha_k) = Z_{1j}(\beta_l) = Z_{1g_\beta(j)}(\beta_{\sigma(l)}) = Z_{0g_\beta(j)}(\alpha_k)$$

thus $Z'_{10}(\alpha'_{k'}) = Z_{00}(\alpha_k) = -Z'_{11}(\beta_l)$ and $h'(\alpha') = Z'_{10}(\alpha'_{k'}) + Z'_{11}(\alpha'_{k'}) = h(\alpha) > 0$.
If $\epsilon(\beta) = -1$, the unique delicate case is when $\beta_{\sigma(l)}$ is the first letter of its word. Then $\eta'_{n'} = \alpha_k$ and $\alpha_k\beta_{\sigma(l)}$ is a new factor and

$$Z'_{0g_\beta(1)}(\eta'_{n'}) = Z'_{\iota_\beta(10)}(\alpha_k) = Z_{\iota_\beta(10)}(\beta_{\sigma(l)}) = 0$$

5. Finally $Z'_{\iota_\beta(ij)(\alpha_k)}$ becomes positive since we add heights to potentially negative zip values.

Let us now show that $Z'_{11}(\beta_l) < 0$. Notice that by definition we have $Z'_{11}(\beta_l) = -Z_{00}(\alpha_k) = -Z_{\iota_\alpha(00)}(\alpha_{\sigma(k)})$ and $\alpha \neq \beta$. If α does not appear as a letter at the beginning of ν or ω then condition 6 implies the inequality and its values clearly cover the whole set. If α is the first letter of ν, then $\delta(\alpha) = -1$ and $Z_{00}(\alpha_{\sigma(k)}) = Z_{f_\alpha(0)1}(\alpha_k) = 0$. If α is the first letter of ω, then $\epsilon(\alpha) = 1$ and $Z_{01}(\alpha_{\sigma(k)}) = Z_{f_\alpha(0)1}(\alpha_k) = 0$. In these two cases, the equality does not affect the value of $Z_{00}(\alpha_k)$ which is then again positive by condition 6.

One can express explicitly the inverse of the affine map Θ_e.

$$\begin{aligned} h(\alpha) &= Z'_{\iota_\beta(01)}(\alpha_k) + Z'_{\iota_\beta(11)}(\alpha_k) & \qquad Z_{10}(\beta_l) &= Z'_{\iota_\beta(10)}(\alpha_k) \\ h(\beta) &= Z'_{10}(\beta_l) + Z'_{11}(\beta_l) & \qquad Z_{11}(\beta_l) &= Z'_{\iota_\beta(10)}(\alpha_k) - h(\alpha) \end{aligned}$$

$$\begin{aligned} Z_{00}(\alpha_k) &= Z'_{\iota_\beta(00)}(\alpha_k) - h(\beta) \\ Z_{01}(\alpha_k) &= Z'_{\iota_\beta(01)}(\alpha_k) \\ Z_{10}(\alpha_k) &= Z'_{\iota_\beta(10)}(\alpha_k) - h(\beta) \\ Z_{10}(\alpha_k) &= Z'_{\iota_\beta(10)}(\alpha_k) \end{aligned}$$

We let the reader check that this sends $\mathcal{Z}^0_{m'}$ to a subset of $\mathcal{Z}_m$. □

Definition 4.5. *Let*

$$\Delta_m^0 := \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha < \lambda_\beta\}$$

and

$$\Delta_m^1 := \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha > \lambda_\beta\}$$

where α, β *denote the last letters of the two words of* m. *These length parameter spaces correspond to bottom or top winning respectively.*

Proposition 4.6. *For any signed matching* m*, the extended Rauzy–Veech induction restricted to* $\big(\{m\} \times \Delta_m^0 \times \mathcal{Z}_m\big)$ *and* $\big(\{m\} \times \Delta_m^1 \times \mathcal{Z}_m\big)$ *respectively defines two invertible linear maps*

$$R_0(m) : \{m\} \times \Delta_m^0 \times \mathcal{Z}_m \longrightarrow \{m_0\} \times \Delta_{m_0} \times \mathcal{Z}_{m_0}^0$$

and

$$R_1(m) : \{m\} \times \Delta_m^1 \times \mathcal{Z}_m \longrightarrow \{m_1\} \times \Delta_{m_1} \times \mathcal{Z}_{m_1}^1.$$

Corollary 4.7. *The extended Rauzy map* $\widehat{\mathcal{R}}_n$ *is a bijection on its image.*

Proof. Assume bottom has won, one can revert the action on the sign matching by finding the losing letter. It must be next to the twin label of the winner. The value of ϵ being unchanged for the winner (which is the last letter on top). And it is the same if top has won.

One can then construct the inverse of $\widehat{\mathcal{R}}_n$. We decide which letter has won by looking at the sign of $Z_{11}(\beta_l)$ and take the inverse by the corresponding map in the previous proposition. □

4.1.3 Non-emptyness

One sensible thing to check here is that the space of possible zip functions is not empty.

In the case of interval exchanges, Masur and Veech introduced a clever explicit construction with integer zip heights defined using the permutation, which are positive when the permutation is irreducible. In [BL09], the authors adapt this trick to the case of linear involutions (without flips). They have to be very smart and work around some other difficulties coming from the fact that irreducibility property is not as neat as in the interval exchange case.

In the end of this section, we will prove that zippered rectangle define a natural extension. Thus an infinite path in the past must define uniquely zip functions. This motivates a construction of a zip function using the induction.

It is easy to find zip function satisfying all properties but 3. The induction then preserves these properties and more importantly gets us closer to satisfying 3.

Proposition 4.8. *After one step of induction, property 3 gets satisfied for the new singularity, i.e. between the twin winning letter and the losing letter and preserves this property on other singularities.*

Thus one only have to find a path along which all singularities lose at least once. This must be the case for a finite power of a positive path since the corresponding length vector goes to zero. This implies the following lemma.

Lemma 4.9. *For every irreducible signed matching, there exists a zip function.*

4.1.4 Minimality

By the interpretation of zippered rectangles as suspension flow of the linear involution and Rauzy–Veech induction as a first return map, the suspension time above the losing interval should be added the one above the winning interval. This property is easily checked using the defining formulas.

Proposition 4.10. *Let h and h' in $\mathbb{R}^{\mathcal{A}}$ be the height function and its image by the extended Rauzy–Veech induction. If M_e is the matrix of the induction defined in the win-lose formalism, then*

$$h' = M_e^t h.$$

Consider H_m the vector subspace of $\mathbb{R}^{\mathcal{A}_m}$ satisfying equalities in conditions 1–4. Consider the canonical basis of $\mathbb{R}^{\mathcal{A}_m}$ and extract a basis for H_m which be endowed with a canonical scalar product. The space $\mathcal{Z}_m$ is a subcone of H_m where all but one coordinates in the basis are positive.

From the definition equations we have the following decomposition of the linear map associated to the extended Rauzy–Veech induction.

Proposition 4.11. *Let m be a signed matching and m' its image throught R_i with $i \in \{0, 1\}$. There exists an orthogonal map $U : H_m \mapsto H_{m'}$ and a vector $v(h) \in H_m$, whose coordinates are positive linear combinations of heights, such that $\widehat{\mathcal{R}}_n$ acts on H_m as $U + v(h)$.*

In particular, for a given zip fonction Z, the image zip function $Z^{(k)}$ after k steps of induction, can be expressed as $Z^{(k)} = U^{(k)}Z + v^{(k)}$ where $U^{(k)}$ is an orthogonal matrix and $v^{(k)}$ is a linear combination of heights along the path $h(0), h(1), \dots, h(k-1)$.

By Proposition 4.10, the scalar product between λ and h is preserved. It correspond geometrically to the area of the surface. One can thus consider the induction restricted to the subspace of parameters where the scalar product $\langle \lambda, h \rangle = 1$. On this subspace we have the following key lemma.

Lemma 4.12. *The extended Rauzy–Veech induction restricted to parameters where $\langle \lambda, h \rangle = 1$ is the natural extension of Rauzy–Veech induction.*

Proof. We proved in Corollary 4.7 that the extended Rauzy–Veech induction is a bijection. The set of parameters with $\langle \lambda, h \rangle = 1$ projects surjectively to all possible lengths. It remains to prove that the coding in the past of a given zip function Z determines it uniquely for almost all paths. We follow Bufetov's scheme of proof in [Buf06].

Consider a path $\gamma_* = e_1 \dots e_n$ in the Rauzy diagram for which the associated path matrix $A = M_{e_1} \dots M_{e_n}$ is positive. Let us consider the unique invariant measure equivalent to Lebesgue. For almost every lengths and zip parameters, in the corresponding bi-infinite path

$$\gamma = \dots \gamma_{-k}\gamma_{-k+1} \dots \gamma_{-1}\gamma_0\gamma_1 \dots \gamma_{k-1}\gamma_k \dots$$

in the Rauzy diagram, the positive path γ_* appears infinitely many times in the future and the past of the coding (see e.g. Proposition 2.8 in [Fou24a]). Moreover, by Proposition 4.10, the action of Rauzy–Veech induction on the heights of zippered rectangles is given by the transpose of paths matrices. Thus if $M_{(-k)}$ is the matrix induced by win-lose induction along $\gamma_{-k}\dots\gamma_0$, $h \in M^t_{(-k)}\cdot\mathbb{R}^{\mathcal{A}}_+$. As A is uniformly contracting Hilbert distance on the projectivized parameter space,

$$\bigcap_{k=0}^{\infty} M^t_{(-k)}\cdot\mathbb{R}^{\mathcal{A}}_+ = \mathbb{R}_+\cdot h_\infty$$

is reduced to one ray and thus the heights h is defined uniquely up to a multiplicative constant by the past coding $\dots\gamma_{-k}\gamma_{-k+1}\dots\gamma_0$ (see e.g. section 3.1 in [Fou24a]). The condition on scalar product then defines it uniquely.

Moreover, coefficients of $M_{(-k)}$ go to infinity as k goes to infinity. The height of zippered rectangles at step $-k$ satisfies $M^t_{(-k)}h(-k) = h$ thus $h(-k)\to 0$ and $Z^{(-k)}\to 0$ since its coefficients are bounded by the heights.

Using the previous proposition, we see that the zip functions $Z^{(-k)}$ such that after k steps of Rauzy–Veech induction we get that Z can be expressed as

$$Z = U^{(-k)}Z^{(-k)} + v^{(-k)}$$

with $U^{(-k)}$ orthogonal and $v(-k)$ depending only on heigths $h(-k), h(-k+1),\dots,h(-1)$ which themselves are determined by the past coding. Hence

$$Z = \lim_{n\to\infty} v^{(-n)}\left(h(-n), h(-n+1),\dots,h(-1)\right)$$

and thus Z only depends on the past coding for almost every points. □

Remark 4.13. *The Lebesgue measure on the space of zippered rectangles is clearly invariant by extended Rauzy–Veech induction. This measure can be introduced intrinsically directly on strata of Abelian or quadratic differentials and is commonly called Masur–Veech measure (see [Zor06] for more background).*

4.2 Teichmüller flow

By classical results of the theory, Teichmüller flow is finite-to-one semi-conjugated to the extension of Rauzy–Veech induction to zippered rectangles. According to Lemma 4.12, it corresponds to natural extension of Rauzy map.

Thus, the ergodic ergodic properties of the canonical suspension flow for Rauzy–Veech induction ergodic properties for the Teichmüller flow on connected components of strata of abelian or quadratic differentials.

In the following we show that Masur–Veech measure is the unique measure of maximal entropy for these flows as well as Central Limit Theorem. These results where known in the case of Abelian differentials [BG11], [Buf06], . This also implies a common proof for exponential mixing which where proved by [AGY06] in the case of Abelian differential and [AR12] for quadratic differentials.

4.2.1 Abelian differentials

Let $\mathcal{R}$ be a connected component of Rauzy diagram. Let $\widetilde{T}:\widetilde{\Delta}_{\mathcal{R}}\to\widetilde{\Delta}_{\mathcal{R}}$ be the natural extension of Rauzy–Veech induction, where $\widetilde{\Delta}_{\mathcal{R}}$ is the associated space of zippered rectangles. Let ψ_t be the canonical suspension flow associated to $\widetilde{T}$. And g_t the Teichmüller flow on the stratum.

Theorem (Veech)**.** *For every connected component of a normalized strata of Abelian differentials $\mathcal{H}_1$, there exists a connected component in Rauzy diagram $\mathcal{R}$ and a finite-to-one map $\pi_{\mathcal{R}}:\widetilde{\Delta}_{\mathcal{R}}\mapsto\mathcal{H}_1$ such that $\pi_{\mathcal{R}}\circ\psi_t = g_t\circ\pi_{\mathcal{R}}$.*

The Lebesgue measure induces by pull back an invariant measure on $\widetilde{\Delta}_{\mathcal{R}}$ and integrating along fibers of the suspension an invariant measure μ on $\Delta_{\mathcal{R}}$ absolutely continuous with respect to Lebesgue measure. As a consequence of Theorem A we get the following.

Corollary 4.14. *The Masur–Veech measure on normalized strata of Abelian differentials is finite it is the unique measure of maximal entropy for the Teichmüller flow and its entropy is equal to* $|\mathcal{A}|$.

4.2.2 Quadratic differentials

The family of linear forms defined by $\delta_{(\nu,\omega,\epsilon)}(\lambda) = \lambda(\nu) - \lambda(\omega)$ is invariant with respect to the induction. As we saw in Proposition 3.8, the kernel of this linear form intersects the positive cone if and only if the linear involution is balanced.

Proposition 5.2 in [BL09] and results of Section 3 in [Zor08] imply the following connection with Teichmüller flow.

Theorem (Boissy–Lanneau, Zorich)**.** *For every connected component of a normalized strata of quadratic differentials* $\mathcal{Q}_1$*, there exists a connected component in Rauzy diagram* $\mathcal{R}$ *and a finite-to-one map* $\pi_{\mathcal{R}} : \widetilde{\Delta}_{\mathcal{R}} \mapsto \mathcal{Q}_1$ *such that* $\pi_{\mathcal{R}} \circ \psi_t = g_t \circ \pi_{\mathcal{R}}$.

By Proposition 3.28 and Theorem A' we then have the following.

Corollary 4.15. *The Masur–Veech measure on strata of quadratic differentials is the unique measure of maximal entropy for the Teichmüller flow and its entropy is equal to* $|\mathcal{A}| - 1$.

4.2.3 Other consequences

Besides from results on measures of maximal entropy, we show in this work that the Teichmüller flow satisfies some exponential tail property. Which implies in particular a Central Limit theorem for Teichmüller flow. Generalizing result of [Buf06] to quadratic differentials in a unified proof with the abelian differential case.

To use Theorem C on a connected component of a normalized stratum, we consider Hölder function in the sense of Veech a function $f : \mathcal{Q}_1 \to \mathbb{R}$ such that there exists a Hölder function on

Theorem 4.16. *Let* $p > 2$ *and let* $f \in L^p(\mathcal{C})$ *with respect to Masur–Veech measure, Hölder in the sense of Veech, and satisfying* $\int f \, \mathrm{d}\mu = 0$ *. Assume that there does not exist* $g \in L^2(\mathcal{Q}_1)$ *such that* $f = X_t g$*, where* X_t *is the derivative in the direction of the Teichmüller flow* g_t*. Then there exists a positive constant* σ_f *such that*

$$\frac{1}{\sqrt{|L|}} \int_0^L f \circ \Phi_t \, dt \xrightarrow{d} \mathcal{N}(0, \sigma_f) \quad \text{as } |L| \to \infty.$$

Where the convergence is in distribution to a normal law of variance σ_f.

Moreover, this shows exponential tail in both cases which is the key ingredient in [AGY06] (see Theorem 4.7) and [AR12] (see Theorem 6.7) to apply Dolgopyat spectrum estimates and show exponential mixing. Hopefully this unified proof brings new insights to this property.

4.2.4 Flipped cases

In the flipped cases, we have seen in Theorem 3.33 that the suspension of generic IET or LI contains a Möbius strip. This led Danthony–Nogueira to prove that a generic measured foliation on a non-orientable surface contains a one-sided closed leaf. In particular when trying to generalize the Teichmüller flow to such non-orientable translation surfaces, this implies that it goes to infinity at almost every point (since one would contract that closed loop and have a degenerating systole). Nonetheless, the Teichmüller flow should have an interesting dynamics on the — zero measure — remaining part of the moduli space of such structures.

The canonical flow defined in the present work on irreducible components of the Rauzy diagram for IET and LI with flips should be conjugated to this restriction of the Teichmüller flow.

Nomenclature

$\mathcal{G}_n$, G	graph defining Rauzy–Veech induction as a win-lose induction.	(p.15)
$\mathcal{G}_n^\phi$, G^ϕ	subgraph of $\mathcal{G}_n$ composed of balanced signed matching.	(p.16)
$\Sigma(\mathcal{G}_n)$	set of signed matching on n letters.	(p.13)
$\Sigma^u(\mathcal{G}_n)$	subset of unbalanced signed matching on n letters.	(p.13)
$\Sigma^0(\mathcal{G}_n)$	subset of signed matching (ν, ω, ϵ) that are either unbalanced or for which the last letters of ν and ω are equal.	(p.13)
F	an irreducible strongly connected component of $\mathcal{G}_n^\phi$.	(p.18)

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