Vector memory random walks and dynamics of simplicial systems

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Abstract

Motivated by the study of generalizations of continued fraction algorithms, we introduce a novel law of random walk on directed graphs, characterized by an infinite memory encoded through the action of finite paths in the graph on a finite-dimensional vector space. When the action is linear, it may have a geometric model on simplices which associates a deterministic dynamical system on the product of a simplex with vertices of the graph.

We establish a criterion on the graph ensuring almost sure recurrence of the random walk and strong ergodic properties for an associated geometric model. This framework contains in particular multidimensional continued fraction algorithms and Rauzy–Veech inductions and offers a unified approach to describing their dynamical behaviors. It not only proves new results on them but also offers fresh insights into their fundamental similarities and differences.

Moreover, our geometric framework serves as a powerful tool for analyzing associated fractal sets. We provide an explicit upper bound on the Hausdorff dimensions of such fractals, which applies broadly across these objects.

As a direct application, we provide in two companion works [Fou25b] and [Fou25a] a new unified proof as well as new results on ergodic properties for classical multidimensional continued fraction algorithms and for generalizations of Rauzy–Veech induction in the non-orientable case for foliations, surfaces or both. Additionally, we present new bounds on the Hausdorff dimension of Rauzy gasket in all dimensions, fractal sets of vectors with integers in their continued fraction expansion are bounded by a constant, and length parameters of minimal interval exchange or linear involutions with flips.

Contents

To compute the best rational approximations of a real number $0 < x < 1$, the classical approach employs the continued fraction algorithm, also known as the Gauss algorithm. It hinges on the Gauss map

$$G : x \to \left\{ \frac{1}{x} \right\},$$

which maps any positive number to the fractional part of its inverse. The Gauss algorithm then associates to the number x the sequence of positive integers $a_n := \left\lfloor \frac{1}{G^{n-1}(x)} \right\rfloor$ for $n \geq 1$. The resulting sequence of rational numbers

$$\frac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

converges to x as $n \to \infty$ and provides the best approximations of x, satisfying the property that for all integers $a, b > 0$, if $|bx - a| \leq |q_n x - p_n|$, then $b \geq q_n$.

The pursuit to extend this property to the simultaneous approximation of vectors by rational numbers — together with algebraic inquiries into characterizations of elements in finite extensions of $\mathbb{Q}$ — has initiated the theory of Multidimensional Continued Fraction algorithms (MCF). Originating from Jacobi and Poincaré's work in the 19th century, who introduced two distinct generalizations, it led to the introduction of a large variety of algorithms over time. Surprisingly, even the question of convergence on each coordinate of a vector for these algorithms does not have a straightforward answer. This complexity has been vividly demonstrated by Nogueira [Nog95], who showed, two centuries after its introduction, that Poincaré's algorithm fails to converge for almost every vector.

For over three decades, a vibrant community of mathematicians has dedicated their efforts to establishing the dynamical properties of MCF algorithms. These properties include crucial aspects such as convergence [Fis72, Nog95, BL13], as well as broader dynamical characteristics like ergodicity [Sch90, MNS09, BFK15], the construction of invariant measures [AL18, AS17], and estimates on convergence speed through Lyapunov exponents [Lag93, BAG01, FS19].

The first step in generalizing continued fractions is to recognize that the Gauss algorithm arises as the first return map of a more fundamental dynamical system: the projectivization of the Euclidean algorithm. Which operates on the two-dimensional positive cone according to the following rule:

$$F : (x, y) \in \mathbb{R}_+^2 \to \begin{cases} (x - y, y) & \text{if } x > y \\ (x, y - x) & \text{if } x < y \end{cases}.$$

Many, if not all, MCF algorithms (e.g. every case surveyed in [Sch00]) can be analogously described using similar maps in higher dimensions. These maps usually involve subtracting certain coordinates from others based on their order in size.

While MCF algorithms are commonly defined as iterations of a single map[1] on an n-dimensional positive cone, another notable generalization of the Gauss algorithm

[1] or accelerations of such maps, see [Fou25b] for details.

emerges through Rauzy–Veech induction on interval exchange maps. This induction operates across multiple copies of a positive cone associated with each vertex of a combinatorial graph known as a "Rauzy diagram," introduced by Rauzy in [Rau79]. It serves as a foundational concept in Teichmüller dynamics, wielding significant influence over dynamical outcomes concerning translation surfaces and Teichmüller flow.

The exploration of Rauzy–Veech induction has yielded a multitude of significant results in the field. Noteworthy achievements include the establishment of the ergodicity of the Teichmüller flow [Vee82] (also independently proven by Masur [Mas82] using distinct methodologies), the introduction of Lyapunov exponents on translation surfaces by Zorich [Zor96], the demonstration of the existence and uniqueness of a measure of maximal entropy for the Teichmüller flow by Bufetov and Gurevich [BG11], and its exponential mixing rates described by Avila, Gouëzel, and Yoccoz [AGY06]. For a comprehensive overview of these achievements, readers may refer to the survey by Forni and Matheus [FM14].

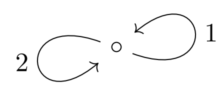

Figure 1: The Rauzy diagram of the Gauss algorithm

The primary objective of this study is to develop a concept analogous to the Rauzy diagram, specifically tailored for general MCF algorithms. This diagram serves as a tool to capture the intricate combinatorial complexities inherent in such algorithms. We introduce what we term as *win-lose induction*, which is defined through a labeled directed graph. In this induction process, each step of the algorithm operates an elementary linear transformation on a positive cone and progresses along a corresponding edge. Consistently with Rauzy–Veech induction, we say the label of the taken edge *loses* and the other labels of outgoing edges *win* at a given step.

After defining win-lose induction in Section 1, we observe that the probability law on paths in the graph induced by vectors picked randomly in the positive cone, with respect to Lebesgue measure, resembles a random walk. This observation leads to the formulation of a general law of random walk, termed as vector memory random walk, where the probability at each step to go through an edge of the other is given by corresponding coordinate of a vector, called distortion vector, which evolves deterministically along the path. To encapsulate the specific properties of win-lose induction, in particular the correlation between its dynamics and properties on the graph, we enumerate four key structural assumptions that will stand in the remaining of the paper.

In a companion work [Fou25b], we explain a general strategy to associate a win-lose induction to a MCF and compute them for most classical MCF algorithms. As an example, the graph for the Brun algorithm in dimension 3 is depicted in Figure 2.

These random walks have the remarkable property that if one goes through a given label, the chance to go through this label again increases in the close future. That reinforcing phenomenon may be the source of non-ergodicity which appears in particular in Poincaré algorithm.

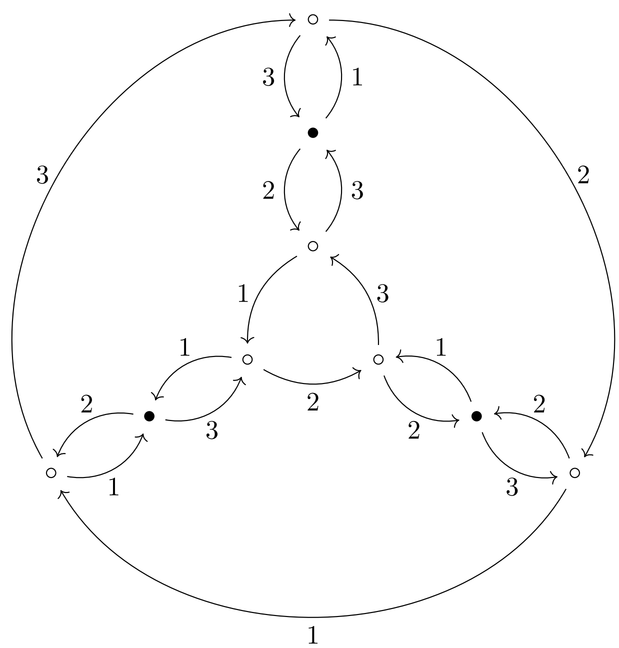

Figure 2: Brun algorithm as a win-lose induction

In Section 2, we use this perspective to develop a criterion directly applicable on the graph that prevents this reinforcing to trap the random walk in a subgraph and thus induces a strong recurrence property on the random walk.

These possible trap subgraphs appear when coordinates of the distortion vector on a subset $\mathcal{L}$ of labels are infinitely larger than others. In that case, any time there is a choice between labels in $\mathcal{L}$ and its complementary, the trajectory will almost surely go along the edge labeled in $\mathcal{L}$. Thus we associate to $\mathcal{L}$ a degenerate subgraph in which we remove such edges not labeled in $\mathcal{L}$.

The corner stone property on the random walk is then to prove that, with high probability, trajectories go out of degenerate subgraphs in a controlled amount of time.

This generalizes a phenomenon, noticed in [Ker85] for Rauzy–Veech induction and later in [CN13] for a family of MCF, that in the degenerate case with labels in $\mathcal{L}$, a letter in $\mathcal{L}$ will always lose eventually to a letter in the complementary set.

We call this the *quick escape property* on the graph.

In the case of Brun algorithm, quick escape property is easy to check. Strongly connected components of degenerate subgraphs of Brun are simple loops on two vertices (see Figure 3). The quick escape property then reduces to showing that almost surely a path leaves such loops in a reasonable time.

This arises from the elementary, yet somewhat magical, Kerckhoff Lemma appearing in [Ker85] and the Appendix of [AGY06] in the case of Rauzy–Veech induction and which extends our setting. We are then reduced to checking essentially that minimal strongly connected components of degenerate subgraphs do not contain a vertex with more than one outgoing edge labeled in $\mathcal{L}$. We called this property on

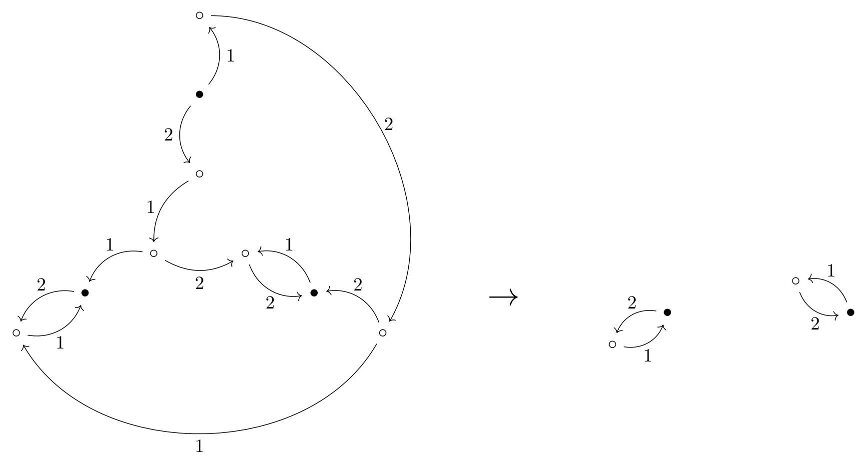

Figure 3: Degenerate subgraph of Brun for $\mathcal{L} = \{1, 2\}$ on the left and its strongly connected components (with multiple vertices) on the right

the graph the *non-degenerating property* and show in Corollary 2.20:

Theorem. *Non-degenerating property implies quick escape property.*

Win-lose inductions not only define a random walk, but have an associated deterministic dynamical system. In Section 3, we introduce the concept of having simplicial models for vector memory random walks, wherein the action of paths on vectors is linear. This gives a deterministic algorithm on spaces constructed as intersection of simplices which mimics this deterministic dynamical system on more general sets. We term them *simplicial systems*, inspired by [Ker85] where the idea of a common setting for Rauzy–Veech induction and renormalization of train tracks was introduced under this name.

While this abstract setting may seem artificial since completely explicit in the case of win-lose induction, it will turn out useful to describe similar dynamics on fractal sets studied later on.

For these types of dynamical systems, we use the knowledge we built in Section 2 on random walks to study their ergodic properties. In particular, we define a suspension of the dynamical system through a roof function r associated to the linear action on paths and show an exponential tail property for it. The main technical tool in this section is thermodynamic formalism via the family of geometric potential function, given by scalar multiples of r.

Theorem. *Every quickly escaping linear memory random walk endowed with a simplicial model has an ergodic measure equivalent to Lebesgue measure and it induces the unique measure of maximal entropy on its canonical suspension.*

Other rich ergodic properties are implied by this theorem such as exponential mixing, central limit theorem and strongly positive recurrence (see Section 3.3.1 and Section 3.5). Moreover the invariant measures can by approximated using periodic orbits with Bowen–Margulis approach. The theorem is also a key step towards counting estimates of periodic orbits.

We check in [Fou25b] the criterion for Brun, Selmer, Jacobi–Perron, Ostrowski algorithms in all dimension and Arnoux–Rauzy–Poincaré algorithm in dimension 3.

And in [Fou25a], for generalizations of Rauzy–Veech induction on non-orientable foliation on orientable and non-orientable surfaces. It provides a unified proof of ergodicity of these algorithms as well as new results on the ergodic properties of their canonical suspension flow (see Section 3.3 for its definition). It proves in particular new results on the ergodic properties of Teichmüller flow, which were only known in the case of Abelian differentials.

Several authors had the intuition that such a setting should exists. There have been partial attempts to introduce global arguments in [Ker85], [CN13] and [MN13] to quote some of them. But they only notice similar properties and scheme of proof for other examples. The latter article for instance ends by conjecturing their methods adapts to Jacobi–Perron. This setting brings a rigorous explanation of this intuition and can be adapted to all the cases known by the author.

It also brings a new perspective on Poincaré algorithm in all dimensions, which are the only examples of classical MCF which are not non-degenerating and for which it is not clear that they have stable degenerate subgraphs (except for the case of dimension 3). Studying ergodicity of Poincaré algorithm reduces in this formalism to compute fine estimates of the time a path in the graph stays in the degenerate subgraph. Moreover, this formalism gives a lot of freedom to introduce new examples of ergodic MCF and find algorithms closer to optimality.

Assume we are given a linear memory random walk on a graph G, and it is endowed with a simplicial model with parameter space $\Delta^\infty(G)$. If it satisfies an extra hypothesis (Leb_*^H) relating the law of random walk with Lebesgue measure of the parameter space, our study on potential functions imply a bound on the Hausdorff dimension of the space.

Theorem. *There exists a unique real value κ for which the Gurrevich–Sarig pressure $P(-\kappa r)$ vanishes. The Hausdorff dimension of the simplicial model is bounded by*

$$\dim_H(\Delta^\infty(G)) \leq d - 1 + \frac{\kappa}{d+1}$$

where d is the dimension of the ambiant simplex.

There are fractal sets naturally emerging from MCF which can be constructed as induced vector memory random walk from a win-lose induction on a graph G to a subgraph F. It also induces a simplicial model for F with parameter space $\Delta^\infty(F, G)$. As a consequence we get the following control on the Hausdorff dimension of the parameter space.

Corollary. *If the induced linear memory random walk from a win-lose induction with base graph G to a subgraph F is quickly escaping, and from every vertex in F there exists a path in G which goes out of F, then the Hausdorff dimension of the simplicial model of $\Delta^\infty(F, G)$ is strictly smaller than the dimension of its ambiant simplex.*

An important example of such sets is the *Rauzy gasket*. It has been primarily introduced by Levitt [Lev93] in connection with the dynamics of partially defined rotations of the circle, and was rediscovered latter by De Leo and Dynnikov [DD09] to study particular classes of examples for Novikov's conjecture in mathematical physics. It was generalized to all dimensions by [AS13] in a word combinatorics approach. More recently it was used in Diophantine geometry [GMR19] to show estimates on the number of integer points on Markoff–Hurwitz varieties.

As an application of this last theorem, we generalize in [Fou25b] the result on the Hausdorff dimension of the Rauzy gasket in [AHS16] to Rauzy gaskets of arbitrary dimensions $\mathcal{G}^d$, as introduced in [AS13]. And provide sharper bounds in all dimension, improving the previous known bounds by [AHS16]. We also prove such results on fractal sets defined by vectors which continued fraction expansions are composed of integers bounded by a constant.

In [Fou25a], we revisit Rauzy–Veech induction generalized to non-orientable foliations and demonstrate that parameters inducing minimal dynamics have Hausdorff dimension strictly smaller than that of the ambiant space.

1 Definitions

We start our study by considering a class of maps inspired by Rauzy–Veech induction, that we call win-lose induction, which serve as a motivation for the introduction of the more general vector memory random walk setting in the following.

1.1 Win-lose induction

Let $G = (V, E)$ denote a graph labeled with an alphabet $\mathcal{A}$, where labeling is carried out by a function $l : E \to \mathcal{A}$ such that, for every $v \in V$, the restriction of l to edges originating from v is injective. We denote an edge going from vertex v to vertex v' as $e : v \to v'$.

Consider $\mathbb{R}_+^{\mathcal{A}} := \left\{x \in \mathbb{R}^{\mathcal{A}} \mid x_\alpha > 0,\ \forall \alpha \in \mathcal{A}\right\}$ and V^0 the set of vertices in V with no outgoing edges. For a vertex v in $V \setminus V^0$, let E_v represent the set of all outgoing edges from v. There is a partition of $\mathbb{R}_+^{\mathcal{A}}$ indexed by edges $e \in E_v$ which tiles are defined by

$$\mathcal{K}^e := \left\{(\lambda_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_{l(e)} < \lambda_\alpha \text{ for all } \alpha \in l(E_v) \text{ and } \alpha \neq l(e)\right\}.$$

Additionally, we associate to each edge $e \in E$ a matrix

$$M_e := \mathrm{Id} + \sum_{\substack{\alpha \in l(E_v) \\ \alpha \neq l(e)}} N(\alpha, l(e)).$$

Where $N(a, b)$ is the elementary matrix with coefficient 1 at row a and column b. Such that $\mathcal{K}^e = M_e \cdot \mathbb{R}_+^{\mathcal{A}}$.

We introduce the *win-lose induction* associated to the base graph G, $\Theta : \left(V \setminus V^0\right) \times \mathbb{R}_+^{\mathcal{A}} \to V \times \mathbb{R}_+^{\mathcal{A}}$, where for all $\lambda \in \mathcal{K}^e$ with $e : v \to v'$, $\Theta(v, \lambda) = (v', \Theta_e(\lambda))$, and

$$\Theta_e : \begin{cases} \mathcal{K}^e & \to & \mathbb{R}_+^{\mathcal{A}} \\ \lambda & \mapsto & M_e^{-1}\lambda \end{cases} .$$

This linear map can be characterized as follows: it compares the coordinates of all edges emanating from a given vertex v on the vector and subtracts the smallest coordinate from the others.

These maps are not well defined on the boundaries of subcones $\mathcal{K}^e$. However, we overlook this detail by referring to the entire space $\left(V \setminus V^0\right) \times \mathbb{R}_+^{\mathcal{A}}$ as its domain since our primary concern lies in their Lebesgue generic dynamics.

In analogy with the standard Rauzy–Veech induction applied to interval exchange transformations (for an introduction, refer to [Yoc10]), we introduce the concepts of loser and winner labels for each edge in the graph, such that at each step, the map substracts the losing label to the winning ones.

Definition. *At a given vertex v with two or more outgoing edges, during the induction step which follows an edge e in the graph, we designate the letter $l(e) \in \mathcal{A}$ as the **loser**. Conversely, for every other edge e' emanating from v, we refer to the letter $l(e') \neq l(e)$ as a **winner**. In both cases, we say that these letters **play** in the induction step.*

Projectivization As such maps commute with multiplication by a positive scalar, it is natural to quotient by such action when studying their dynamics.

For A a real linear space, we denote by $P(A)$ or simply PA the *projectivized* quotient space $A/\sim$ where $\lambda \sim c \cdot \lambda$ for all $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ and c positive real number. Let $\Delta := P\mathbb{R}_+^{\mathcal{A}}$ and $\Delta^e := P\mathcal{K}^e$ As Θ is piecewise linear, it induces a projectivized map

$$T : \left(V \setminus V^0\right) \times \Delta \to V \times \Delta.$$

We use the notations

$$\Delta^+(G) := \left(V \setminus V^0\right) \times \Delta \quad \text{and} \quad \Delta(G) := V \times \Delta.$$

Very often in the literature, these projectivized map appear in a chart where the L^1-norm, $|\lambda| := \sum_{\alpha \in \mathcal{A}} \lambda_\alpha$, is normalized. It is the case for Rauzy–Veech induction, Farey map and other multidimensional continued fraction maps (see [Fou25b] for more details). In these charts, we have for all $\lambda \in \Delta^e$ with $e : v \to v'$, $T(v, \lambda) = (v', T_e(\lambda))$, where $T_e(\lambda) = \dfrac{M_e^{-1}\lambda}{\left|M_e^{-1}\lambda\right|}$. In Figure 4, we represent the action of the map T on subcones in these charts.

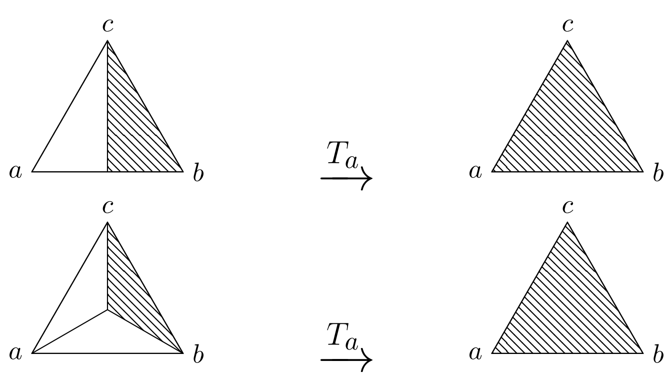

Figure 4: Action of T_a on Δ_a when v has two or three outgoing edges

For a point in $\Delta^+(G)$ all iterates of T are defined until it visits a vertex in V^0. For a vertex $v \in V$, let $\Pi^0(v)$ denote the sets of finite path from v to a vertex in V^0 and $\Pi^\infty(v)$ infinite paths starting at v. By disregarding the preimages of hyperplanes of equality cases in the simplex, on which T is not defined, we can define a coding map which associates to each point of the cone a path in one of these two sets.

Proposition 1.1. *There exists a countable collection $\mathcal{F}$ of codimension-one linear subspaces, outside of which the win-lose induction induces a mapping*

$$c_v : \Delta \setminus \bigcup_{H \in \mathcal{F}} H \to \Pi^0(v) \sqcup \Pi^\infty(v)$$

associating to a vector λ the path followed by the vertex component of $T^n(v, \lambda)$.

To study iterates of T, it will be convenient to consider its restriction to the space $\Delta^\infty(G) := \bigcap_{n\in\mathbb{N}} T^{-n}\Delta(G)$. The image of $\Delta^\infty(G) \cap (\{v\} \times \Delta)$ by c_v — intersected with its domain of definition — is then included in $\Pi^\infty(v)$.

Let $\Pi(v)$ be the set of finite paths in G starting at v — in other terms, the set of finite prefixes of $\Pi^0(v) \sqcup \Pi^\infty(v)$. For $\gamma \in \Pi(v)$, we denote by $\Pi(\gamma)$ the subset of path in $\Pi(v)$ which start with γ.

Proposition 1.2. *For all $v \in V$ and all $\gamma \in \Pi(v)$, $c_v(\Delta^\gamma) = \Pi(\gamma)$ where*

$$\Delta^\gamma := M_\gamma \Delta \quad \textit{with} \quad M_\gamma := M_{e_1} \dots M_{e_n}.$$

Remark. *For clarity in our exposition, we adopt the convention of using variables of the form γ to represent finite paths in $\Pi(v)$, and $\overline{\boldsymbol{\gamma}}$ for paths that are either infinite or terminate at a vertex with no outgoing edges i.e. in $\Pi^0(v) \sqcup \Pi^\infty(v)$. The notation $\overline{\boldsymbol{\gamma}}_n$ refers to the maximal prefix of $\overline{\boldsymbol{\gamma}}$ with length less or equal to n.*

Projective measures A pivotal element for examining the Lebesgue-generic dynamical behavior of a win-lose induction is to analyze its behavior within a stable family of measures equivalent to Lebesgue. The crucial aspect enabling us to derive dynamical outcomes for Lebesgue-generic paths is the tractability of the induction's action on this measure family, through a dual action on a positive vector.

Definition. *Let $q \in \mathbb{R}_+^{\mathcal{A}}$, and let ν_q be the Borel measure on Δ, defined as follows: for any subset $A \subset \Delta$,*

$$\nu_q(A) := \mathrm{Leb}(p^{-1}A \cap \Lambda_q)$$

where $p : \mathbb{R}_+^{\mathcal{A}} \to \mathbb{R}_+^{\mathcal{A}}/\sim\, = \Delta$ is the quotient map and $\Lambda_q = \{v \in \mathbb{R}_+^{\mathcal{A}} \mid \langle q, v\rangle < 1\}$.

A fundamental equality is given by

$$\begin{aligned}\nu_q(M_\gamma A) = \mathrm{Leb}(p^{-1}M_\gamma A \cap \Lambda_q) &= \mathrm{Leb}(p^{-1}A \cap \Lambda_{qM_\gamma}) \\ &= \nu_{qM_\gamma}(A).\end{aligned} \tag{1}$$

The vector q keeps track of the way the measure is changed along the induction; we call it the *distortion vector*. We think of it as a line vector thus matrices M_γ act on its right.

Another fundamental equation arises from an elementary computation that can be found, for example, in [Vee78] Formula (5.5).

Proposition 1.3 (Veech)**.** *For $v \in V$, $\gamma \in \Pi(v)$ and $q \in \mathbb{R}_{>0}^{\mathcal{A}}$,*

$$\nu_q(\Delta^\gamma) = \frac{1}{n!} \cdot \frac{1}{(qM_\gamma)_1 \dots (qM_\gamma)_n} \tag{2}$$

If v is a vertex and γ is a path starting at v we define the probability measure,

$$\mathbb{P}_q^v(\gamma) = \frac{\nu_q(\Delta^\gamma)}{\nu_q(\Delta)}.$$

According to Formula (2),

$$\mathbb{P}_q^v(\gamma) = \frac{N(q)}{N(qM_\gamma)} \quad \textit{where} \quad N(q) = \prod_{a\in\mathcal{A}} q_a. \tag{3}$$

Proposition 1.4. *Let $e : v \to v' \in E$ labeled by α, then*

$$\mathbb{P}_q^v(e) = \frac{q_\alpha}{(qM_e)_\alpha} = \frac{q_\alpha}{\sum\limits_{\beta \in l(E_v)} q_\beta}.$$

Proof. Just notice that for all $\beta \neq l(e)$, $(qM_e)_\beta = q_\beta$. □

Definition 1.5. *We associate a law of random walk with any graph defining a win-lose induction. For each vertex $v \in V$ and distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$, let*

- *the sample space $\Pi^0(v) \sqcup \Pi^\infty(v)$,*
- *the set of events given by the σ-algebra generated by sets*

$$\left\{\gamma_s \cdot \overline{\boldsymbol{\gamma}}_t \mid \overline{\boldsymbol{\gamma}}_t \in \Pi^0(\gamma_s \cdot v) \sqcup \Pi^\infty(\gamma_s \cdot v)\right\}$$

  for $\gamma_s \in \Pi(v)$,
- *the probability law given by the pushforward of $\mathbb{P}_q^v$ by c_v.*

Where the middle point in $\gamma_s \cdot \overline{\boldsymbol{\gamma}}_t$ denotes the concatenation of two paths.

If a path $\gamma \in \Pi(v)$ can be decomposed as $\gamma = \gamma_s \cdot \gamma_t$, where $\gamma_s \in \Pi(v)$ ends at v' and γ_t is a path in $\Pi(v')$, one can define conditional probabilities using Formula (1)

$$\mathbb{P}_q^v(\gamma \mid \gamma_s) = \frac{\nu_q(\Delta^\gamma)}{\nu_q(\Delta^{\gamma_s})} = \frac{\nu_q(M_{\gamma_s}\Delta^{\gamma_t})}{\nu_q(M_{\gamma_s}\Delta)} = \mathbb{P}_{qM_{\gamma_s}}^{v'}(\gamma_t). \tag{4}$$

If Γ_s is a set of finite paths that are not prefixes of one another, we can decompose the probability as follows:

$$\mathbb{P}_q^v\left(\Gamma \cap \Pi(\Gamma_s)\right) = \sum_{\gamma_s \in \Gamma_s} \mathbb{P}_q^{v\cdot\gamma_s}\left(\Gamma \cap \Pi(\gamma_s) \mid \gamma_s\right) \cdot \mathbb{P}_q^v(\gamma_s), \tag{5}$$

where $v \cdot \gamma_s$ denotes the ending vertex of the path γ_s, $\Pi(\gamma_s) := \{\gamma \cdot \gamma_t \mid \gamma_t \in \Pi(v \cdot \gamma_s)\}$, and $\Pi(\Gamma_s) := \bigcup_{\gamma_s \in \Gamma_s} \Pi(\gamma_s)$.

1.2 Vector memory random walk

The property of the random walk associated with a win-lose induction motivates the introduction of a more general stochastic process that we call the *vector memory random walk*, which is *a priori* not induced by a win-lose map or a geometric construction.

Let G be a labeled directed graph with the same properties on labeling as in the previous section, we use the same notations. Assume moreover that for any vertex $v \in V$, there is a right action of $\Pi(v)$ on $\mathbb{R}_+^{\mathcal{A}}$. We define a probability law on $e \in E_v$ associated to a *distortion vector* $q \in \mathbb{R}_+^{\mathcal{A}}$ by

$$\mathbb{P}_q^v(e) = \frac{q_\alpha}{\sum\limits_{\beta \in l(E_v)} q_\beta}$$

where $\alpha = l(e)$ is unique by injectivity assumption on l.

For a given finite path $\gamma \in \Pi(v)$ such that $v \cdot \gamma \notin V^0$, we further define the conditional law for $e \in E_{v\cdot\gamma}$

$$\mathbb{P}_q^v(\gamma \cdot e \mid \gamma) = \mathbb{P}_{q\cdot\gamma}^v(e).$$

These two formulas together define, as in Definition 1.5, a probability law on the measurable space $\Pi^0(v) \sqcup \Pi^\infty(v)$ equipped with the σ-algebra generated by the sets $\{\gamma_s \cdot \overline{\boldsymbol{\gamma}}_t \mid \overline{\boldsymbol{\gamma}}_t \in \Pi^0(\gamma_s \cdot v) \sqcup \Pi^\infty(\gamma_s \cdot v)\}$ for $\gamma_s \in \Pi(v)$.

Remark 1.6. *For convenience and when there is no ambiguity, for an edge e labeled by α, we write the α instead of e in the path.*

In the win-lose induction case, the action is given by $q \cdot \gamma = q \cdot M_\gamma$, and there is additional control on the growth of distortion vectors for winning and losing labels. This control is essential to establish connections between properties of the graph and behaviour of the random walk. We list these essential structural properties satisfied by win-lose inductions.

(H1) For all $q \in \mathbb{R}_+^{\mathcal{A}}$, the coordinates of q are non-decreasing under the action of Γ.

(H2) For all $q \in \mathbb{R}_+^{\mathcal{A}}$, all label $\alpha \in \mathcal{A}$, and all infinite path $\overline{\boldsymbol{\gamma}}$ in the graph such that α loses infinitely many times, we have

$$(q \cdot \overline{\boldsymbol{\gamma}}_n)_\alpha \longrightarrow \infty.$$

(H3) There exist $A > 1$ such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$ and for every edge e and every label $\alpha \in \mathcal{A}$,

$$(q \cdot e)_\alpha \leq A \cdot \max q.$$

These hypotheses ensure regularity in the growth of the memory vector after each step of the path action. In the win-lose induction case, for an edge $e : v \to v'$ labeled α the distortion on the set of winning letters $l(E_v) \setminus \alpha$ is added to distortion on the losing one, thus implying a key balancing phenomenon.

For general random walks, we define winners as follows. Which ensures a similar balancing phenomenon with respect to these winning labels.

Definition 1.7. *For labels $\alpha, \beta \in \mathcal{A}$, we say that β wins against α along a path γ in the graph if there exists $B \geq 1$ such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$,*

$$q_\alpha + \frac{1}{B} \cdot q_\beta \leq (q \cdot \gamma)_\alpha.$$

The following property claims that, along an edge $e : v \to v'$ labeled α, labels in $l(E_v) \setminus \alpha$ win against α.

(H4) There exists $B \geq 1$ such that for all edge $e : v \to v'$ labeled α and all $q \in \mathbb{R}_+^{\mathcal{A}}$,

$$q_\alpha + \sum_{\substack{\beta \in l(E_v) \\ \beta \neq \alpha}} \frac{1}{B} \cdot q_\beta \leq (q \cdot \alpha)_\alpha.$$

(K) For all vertex v and every label $\alpha \in l(E_v)$, the following two properties hold.

- **a.** For all $q \in \mathbb{R}_+^{\mathcal{A}}$ and all labels $\delta \neq \alpha$, $(q \cdot \alpha)_\delta = q_\delta$.
- **b.** There exists $M > 1$ such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$, $(q \cdot \alpha)_\alpha \leq q_\alpha + M \cdot \sum_{\substack{\beta \in l(E_v) \\ \beta \neq \alpha}} q_\beta$.

Remark 1.8. *Assumption (K)-**b.**, implies that for an edge $e : v \to v'$ labeled α, no letters outside of $l(E_v)$ can win against α, hence together with (H4) the set of winning labels against α is exactly $l(E_v) \setminus \alpha$.*

*Also notice that assumption (K)-**a.** together with (H4) implies (H1). But our first result will only require hypothesis (H1-4) so we prefer to state it independently.*

Assumption (K) is satisfied for $M = 1$ in the win-lose induction case. It will only be used later on, starting in Section 2.2, to prove Kerckhoff lemma (Lemma 2.12). This lemma, however, is derived from the following upper bound on probability.

Proposition 1.9. *Assumption (K)-**b**. implies that there exists $\sigma > 0$ such that for all vertex v, every $q \in \mathbb{R}_+^{\mathcal{A}}$ and all $\alpha \in l(E_v)$, $\mathbb{P}_q^v(\alpha) \leq \left(\frac{q_\alpha}{(q\cdot\alpha)_\alpha}\right)^\sigma$.*

Proof. For $q \in \mathbb{R}_+^{\mathcal{A}}$ and an edge $e : v \to v'$ labeled α as in the proposition, let us define $Q = \sum_{\beta \in l(E_v)} q_\beta$ and $M' = M \cdot (Q - q_\alpha)$. We have

$$\frac{q_\alpha}{(q\cdot\alpha)_\alpha} \geq \frac{q_\alpha}{q_\alpha + M'} \geq \frac{q_\alpha}{Q + M'}.$$

As $\mathbb{P}_q^v(\alpha) = \frac{q_\alpha}{Q}$, we aim to prove,

$$\begin{aligned} \frac{q_\alpha}{Q} \leq \left(\frac{q_\alpha}{Q+M'}\right)^\sigma &\iff (1-\sigma)\cdot \ln\frac{q_\alpha}{Q} \leq \sigma \cdot \ln \frac{1}{1+\frac{M'}{Q}} \\ &\iff \ln\left(1+\frac{M'}{Q}\right) \leq \frac{1-\sigma}{\sigma}\cdot \ln\frac{Q}{q_\alpha}. \end{aligned}$$

Observe that $\ln\left(1+\frac{M'}{Q}\right) \leq \frac{M'}{Q}$ and $1 - \frac{q_\alpha}{Q} \leq \ln\frac{Q}{q_\alpha}$.
Hence, the previous inequality follows from

$$\frac{M'}{Q} \leq \frac{1-\sigma}{\sigma}\cdot\left(1-\frac{q_\alpha}{Q}\right) \iff M\cdot(Q-q_\alpha) \leq \frac{1-\sigma}{\sigma}\cdot(Q-q_\alpha).$$

One simply has to pick $\sigma > 0$ such that $M \leq \frac{1-\sigma}{\sigma}$. □

1.2.1 Stopping times

The probability distribution of paths is highly influenced by the distortion. However, it will be possible to bound the probability that certain well-chosen *stopping times* are smaller than others independently of the distortion.

Definition. *Let $\mathcal{P}$ be a property on finite paths. A stopping time associated with $\mathcal{P}$ is the random variable*

$$T_{\mathcal{P}} : \left\{ \begin{array}{lcl} \Pi^0(v) \sqcup \Pi^\infty(v) & \to & \mathbb{N}\cup\{\infty\} \\ \overline{\boldsymbol{\gamma}} & \mapsto & T_{\mathcal{P}}(\overline{\boldsymbol{\gamma}}) \end{array}\right.$$

where $T_{\mathcal{P}}(\overline{\boldsymbol{\gamma}}) = \min\{n \geq 0 \mid \overline{\boldsymbol{\gamma}}_n$ satisfies $\mathcal{P}\}$ and $\overline{\boldsymbol{\gamma}}_n$ is the maximal prefix of $\overline{\boldsymbol{\gamma}}$ with length less or equal to n.

The map $T_{\mathcal{P}}$ is referred to as a *stopping time* for the vector memory random walk, where the path stops when the property $\mathcal{P}$ is satisfied. In the following, we abbreviate $\mathcal{P}$ instead of $T_{\mathcal{P}}$ for simplicity in formulas.

Proposition 1.10. *For every path $\gamma_s \in \Pi(v)$ and $\overline{\boldsymbol{\gamma}}_t \in \Pi^\infty(\gamma_s \cdot v)$*

$$T_{\mathcal{P}}(\gamma_s \cdot \overline{\boldsymbol{\gamma}}_t) = T_{\gamma_s^{-1}\mathcal{P}}(\overline{\boldsymbol{\gamma}}_t) + |\gamma_s|$$

where $\overline{\boldsymbol{\gamma}}_t$ satisfies $\gamma_s^{-1}\mathcal{P}$ if and only if $\gamma_s \cdot \overline{\boldsymbol{\gamma}}_t$ satisfies $\mathcal{P}$. In particular, when comparing the stopping times corresponding to properties $\mathcal{P}$ and $\mathcal{Q}$, we have, with abbreviated notations,

$$\mathbb{P}_q^v(\mathcal{P} < \mathcal{Q} \mid \gamma_s) = \mathbb{P}_{qM_{\gamma_s}}^v\left(\gamma_s^{-1}\mathcal{P} < \gamma_s^{-1}\mathcal{Q}\right). \tag{6}$$

Remark 1.11. *If $\mathcal{C}$ is a set of paths such that together with its subset of paths starting with γ_s they have non-zero measure, then*

$$\mathbb{P}_q^v\left(\mathcal{P} < \mathcal{Q} \mid \mathcal{C} \text{ and } \gamma_s\right) = \mathbb{P}_{qM_{\gamma_s}}^v\left(\gamma_s^{-1}\mathcal{P} < \gamma_s^{-1}\mathcal{Q} \mid \gamma_s^{-1}\mathcal{C}\right).$$

In particular, if $\gamma_s^{-1}\mathcal{C} = \mathcal{C}$ almost surely, the conditional probability with respect to $\mathcal{C}$ also satisfies relation (6) for all distortion vectors.

1.2.2 Induced random walk

An important source of examples will be given by induced random walks on a subgraph. Consider F a subgraph of a graph G on which a vector memory random walk is defined. By restricting the action of the random walk on G to paths within F, we induce a vector memory random walk on F. For a vertex $v \in F$ and $q \in \mathbb{R}_+^{\mathcal{A}}$, let $\mathbb{P}_q^{v,G}$ denote the probability distribution on paths in G, and $\mathbb{P}_q^{v,F}$ the corresponding distribution on paths in F.

Proposition 1.12. *The induced probability on an edge $e \in F$ is related to the probability on G by*

$$\mathbb{P}_q^{v,F}(e) = \frac{\mathbb{P}_q^{v,G}(e)}{\mathbb{P}_q^{v,G}(\text{edges in } F)}$$

Additionally, for a finite path γ, we have:

$$\mathbb{P}_q^{v,G}(\gamma) \leq \mathbb{P}_q^{v,F}(\gamma)$$

When considering a subgraph, vertices with a unique outgoing edges will often appear. A step of the random walk passing through such vertex is irrelevant from a dynamical and probabilistic perspective. Thus for a graph G having such vertices, we prefer to consider an accelerated random walk which continues as long as the end vertex has a unique outgoing edge. This can be thought of as a random walk on a factor graph.

Definition 1.13. *We define the factor graph $\widehat{G} = (\widehat{V}, \widehat{E})$ of a graph G as follows:*

- *$\widehat{V}$ is the set of branching vertices, i.e. those with two or more outgoing edges.*
- *$\widehat{E}$ is the set of branch paths, i.e. the set of path in G for which the start and end vertices, and only them two, belong to $\widehat{V}$.*

There is an edge between two vertices v and v' in $\widehat{V}$ if and only if they are connected by a path in $\widehat{E}$ that starts at v and ends at v'. Moreover, the label of that edge is the label of the first edge of the path in G.

While the restricted action on a subgraph satisfies assumptions (H1-4) and (K)-**a.**, certain subgraphs, may not preserve the assumption (K)-**b.** in general. Think of a unique outgoing edge with a non-trivial action for instance.

Similarly, the induced action on $\widehat{G}$ also satisfies (H1-4) but may fail to satisfy either of properties in (K). Thus one will have to carefully check Property (K) when associating such induced subgraphs.

Remark 1.14. *Conditions (K) turns out too restrictive for certain applications. We will then introduce later on a weaker version of (K) together with a graph condition that selectively applies the inequality (K)-**b.** to a subset of letters, and enables letters outside of $l(E_v) \setminus \alpha$ to win against α. This refinement will be discussed in Section 2.2.4 and Section 2.2.5.*

2 Generic path in vector memory random walks

In this section, we study generic infinite paths for the law of random walk we have introduced. It describes in particular the Lebesgue generic behaviour of points for win-lose inductions. We assume the following standing assumptions hold for the remaining of the text (except in Section 2.2.4 where a graph with vertices having a unique outgoing edge, inducing (K) to fail, will be discussed).

Standing Assumptions 2.1.

- *The graph is finite and strongly connected.*
- *Every vertex in the graph has at least two outgoing edges.*
- *The action of paths on distortion vectors satisfies hypothesis (H1-4) and (K).*

Strong connectivity can always be assumed for a vector memory random walk by considering the strongly connected component of the starting vertex and consider its induced random walk as in Section 1.2.2. One can also reduce to the second property by considering a factored random walk, but one should be careful that Property (K) may fail on the induced system doing so; this will be discussed in Section 2.2.4.

Assumption (K) is only used after the first subsection to prove Kerckhoff lemma (Lemma 2.12). This lemma is the key tool for what follows but the assumption itself is not further used after its proof.

The cornerstone of our understanding of a generic path behavior lies in the phenomenon that *balanced distortion vectors*, as defined in the next subsection, induce equivalent measures. Hence, our aim is to demonstrate that, independently of the starting distortion vector, a path will almost surely reach a balanced state after a reasonable number of steps. The influence of past steps on the random walk then dissipates at such time.

We first define the *quick escape property* on the random walk, which triggers this balancing phenomenon. Subsequently, we formulate a graph criterion that induces this property and will be easier to manipulate and check.

2.1 Quick escape property

We start by introducing some useful properties for which we compare the stopping times.

Let $\mathcal{J}^\tau$ be the property of a finite path γ along which the distortion vector has *jumped* by a factor $\tau > 0$ *i.e.* it satisfies

$$\max\{q \cdot \gamma\} \geq \tau \max q$$

where the maximum is taken on all the coordinates of the vector.

Remark 2.2. *By the second item of the Standing Assumptions 2.1 and (H2), for all $\tau > 0$, $v \in V$ and all $q \in \mathbb{R}_+^{\mathcal{A}}$, $\mathbb{P}_q^v(\mathcal{J}^\tau = \infty) = 0$.*

A key property to consider for the distortion vector is the balance between its coordinates, as defined below.

Definition. *For $\mathcal{L} \subset \mathcal{A}$ and $K > 1$, a distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$ is called $(\mathcal{L}, K)$-balanced if and only if*

$$\max_{\mathcal{A}} q < K \min_{\mathcal{L}} q$$

and

$$\max_{\mathcal{A}\setminus\mathcal{L}} q \leq \min_{\mathcal{L}} q.$$

When $\mathcal{L} = \mathcal{A}$, we simply say the vector is K-balanced.

Proposition 2.3. *If q is $(\mathcal{L}, K)$-balanced, then after a letter in $\mathcal{L}$ loses, there exists a set $\mathcal{L}'$ containing $\mathcal{L}'$ such that the updated distortion vector remains $(\mathcal{L}', A \cdot K)$-balanced.*

Proof. By assumption (H3), for every edge E, the new distortion vector $q' = q \cdot E$ satisfies the following for every letter $\alpha \in \mathcal{A}$:

$$q'_\alpha \leq A \cdot \max_{\mathcal{A}} q \leq A \cdot K \cdot \min_{\mathcal{L}} q.$$

Let $\mathcal{L}'$ be the set of labels α such that $q'_\alpha \geq \min_{\mathcal{L}} q$. Then, $\mathcal{L} \subseteq \mathcal{L}'$, and we have $\max_{\mathcal{A}\setminus\mathcal{L}'} q' < \min_{\mathcal{L}} q \leq \min_{\mathcal{L}'} q'$. □

This definition is very useful due to the fact that it implies a lower bound on the probability that an edge labeled in $\mathcal{L}$ loses.

Proposition 2.4. *Let $v \in V$ and q be a $(\mathcal{L}, K)$-balanced distortion vector, then for all edge e starting at v and labeled in $\mathcal{L}$*

$$\mathbb{P}_q^v(e) \geq \frac{1}{A \cdot K}$$

and all n-path γ of edges labeled in $\mathcal{L}$,

$$\mathbb{P}_q^v(\gamma) \geq \left(\frac{1}{A^n \cdot K}\right)^n.$$

Let $\mathcal{S}_{\mathcal{L}}$ be the property of a finite path γ for which

$$\max_{\mathcal{A}\setminus\mathcal{L}} \{q \cdot \gamma\} \geq \frac{1}{B} \cdot \min_{\mathcal{L}} q.$$

Where B is the constant in assumption (H4). The stopping time corresponds to when the distortion on a coordinate outside of the subset $\mathcal{L}$ reaches the size, up to a factor B, of the initial distortion on coordinates in $\mathcal{L}$.

Remark 2.5. *If we start with a $(\mathcal{L}, K)$-balanced distortion vector, the set $\mathcal{L}$ contains the $|\mathcal{L}|$ largest coordinates. Thus, when a letter β in $\mathcal{L}$ wins against a letter α not in $\mathcal{L}$, we have $q'_\alpha \geq \frac{1}{B} q_\beta$ which implies property $\mathcal{S}_{\mathcal{L}}$.*

Definition (Quick escape property)**.** *We say a vector memory random walk is quickly escaping if for all non-empty subset $\mathcal{L} \subsetneq \mathcal{A}$ and all $K > 1$ there exist $\tau > 1$ and $\epsilon > 0$ such that for all vertex $v \in V$ and all $(\mathcal{L}, K)$-balanced distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$*

$$\mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}} < \mathcal{J}^\tau) > \epsilon.$$

Let $\mathcal{M}_{\mathcal{L}}^\kappa$ denote the property of a finite path γ for which

$$\kappa \cdot \min_{\mathcal{L}} \{q \cdot \gamma\} \geq \max_{\mathcal{A}} q.$$

When $\mathcal{L} = \mathcal{A}$, we refer to this property simply as $\mathcal{M}^\kappa$.

Lemma 2.6. *If a vector memory random walk is quickly escaping, then there exist constants $\tau > 1$, $\kappa > 0$, and $\epsilon > 0$ such that for every vertex $v \in V$ and every distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$,*

$$\mathbb{P}_q^v(\mathcal{M}^\kappa < \mathcal{J}^\tau) > \epsilon. \tag{7}$$

Proof. We prove this by induction on n. Specifically, we show that for each n, there exist constants $\tau_n > 1$, $\kappa_n > 0$, and $\epsilon_n > 0$ such that for every vertex v and every distortion vector q, there exists a subset $\mathcal{L} \subset \mathcal{A}$ with $|\mathcal{L}| = n$ that satisfies:

$$\mathbb{P}_q^v(\mathcal{M}_{\mathcal{L}}^{\kappa_n} < \mathcal{J}^{\tau_n}) > \epsilon_n.$$

For $n = 1$, we can take $\mathcal{L}$ to be the singleton containing the coordinate of q with the largest value. Then for $\kappa \geq 1$ and $\tau > 1$, $\mathcal{M}^\kappa$ is satisfied at time 0 whereas $\mathcal{J}^\tau$ is not.

Assume the property holds for some $n \geq 1$. Consider a vertex v and the corresponding set $\mathcal{L}$ from the induction hypothesis. With probability greater than ϵ_n, the random walk starts with a path γ such that the distortion vector satisfies $\kappa_n \cdot \min_{\mathcal{L}}(q \cdot \gamma) \geq \max_{\mathcal{A}} q$ and $\max_{\mathcal{A}}(q \cdot \gamma) < \tau_n \cdot \max_{\mathcal{A}} q$. Thus, the updated distortion vector $q' = q \cdot \gamma$ is $(\mathcal{L}', \frac{\kappa_n}{\tau_n})$-balanced for some set $\mathcal{L}'$ containing $\mathcal{L}$.

If $\mathcal{L}' \neq \mathcal{L}$, the argument is straightforward. Otherwise, the quick escape property of the walk implies that there exist $\tau > 1$ and $\epsilon > 0$ such that, with probability greater than ϵ, the random walk starts with a path γ' where a letter $\alpha \in \mathcal{L}$ and a letter $\beta \notin \mathcal{L}$ satisfy $(q' \cdot \gamma')_\beta \geq \frac{1}{B} \cdot q'_\alpha$ before q' jumps by τ. Consequently, $(q' \cdot \gamma')_\beta \geq \frac{1}{B} \cdot \frac{\tau_n}{\kappa_n} \cdot \max_{\mathcal{A}} q'$, ensuring that $\mathcal{M}_{\mathcal{L}'}^{\kappa_{n+1}}$ holds with $\kappa_{n+1} = \kappa_n \cdot \frac{B}{\tau_n}$, $\tau_{n+1} = \tau_n \cdot \tau$, and $\mathcal{L}' = \mathcal{L} \cup \{\beta\}$ before $\mathcal{J}^{\tau_{n+1}}$.

For at least $\frac{1}{|\mathcal{A}|}$ of the paths, the sets $\mathcal{L}'$ share a common letter β. Using the chain rule (5), we establish the induction step for $n + 1$ with $\epsilon_{n+1} = \epsilon_n \cdot \frac{\epsilon}{|\mathcal{A}|}$. □

Let $\mathcal{E}_{\gamma^*}$ denote the property that γ^* as a suffix of the path. In other words, γ satisfies $\mathcal{E}_{\gamma^*}$ if it can be factored as $\gamma = \gamma_0 \cdot \gamma^*$ for some finite path γ_0.

Corollary 2.7. *If a vector memory random walk is quickly escaping then, for all finite paths γ^* in the graph, there exist $K > 1$ and $\epsilon > 0$ such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$ and all $v \in V$,*

$$\mathbb{P}_q^v(\mathcal{E}_{\gamma^*} < \mathcal{J}^K) > \epsilon.$$

Proof. Let τ, κ and ϵ be as defined in the conclusion of Lemma 2.6. Consider $\overline{\boldsymbol{\gamma}} \in \Pi(v)$ such that $\mathcal{M}(\overline{\boldsymbol{\gamma}}) < \mathcal{J}^\tau(\overline{\boldsymbol{\gamma}})$. Let $n := \mathcal{M}(\overline{\boldsymbol{\gamma}})$ which is almost surely finite by Remark 2.2. Define Γ to be the induced set of finite paths $\overline{\boldsymbol{\gamma}}_n$. This set if such that none of its paths is prefix of another.

We consider in the following a path $\gamma \in \Gamma$. Observe that by device

$$\min(q \cdot \gamma) \geq \frac{1}{\kappa} \cdot \max q > \frac{1}{\tau \cdot \kappa} \cdot \max q \cdot \gamma.$$

Thus the distortion vector $q' := q \cdot \gamma$ is τ'-balanced for $\tau' = \tau \cdot \kappa$.

Let v_0 be the starting vertex of γ^*. By the strong connectivity hypothesis, there exists a path $\gamma_0 \in \Pi(v \cdot \gamma)$, with a length bounded by the diameter of the graph, from the vertex $v' := v \cdot \gamma$ to v_0. By Proposition 2.4, there exists ϵ', depending only on τ, κ and the lengths of γ and γ_0, such that

$$\mathbb{P}_{q'}^{v'}(\gamma_0 \cdot \gamma^*) > \epsilon'.$$

For sufficiently large K, independent of γ, any such path $\gamma_0 \cdot \gamma^*$ does not jump by a factor of K on a τ'-balanced distortion vector. Hence,

$$\mathbb{P}_{q'}^{v'}\left(\mathcal{E}_{\gamma^*} < \mathcal{J}^K\right) \geq \mathbb{P}_{q'}^{v'}(\gamma_0 \cdot \gamma^*).$$

If a path γ_t satisfies $\mathcal{E}_{\gamma^*}$, then so does $\gamma \cdot \gamma_t$ and if the path $\gamma \cdot \gamma_t$ jumps by τK, then γ_t must jump by at least K, since γ jumps by less than τ. Hence, with notation of Proposition 1.10, we have $\gamma^{-1}\mathcal{E}_{\gamma^*} \leq \mathcal{E}_{\gamma^*}$ and $\mathcal{J}^K \leq \gamma^{-1}\mathcal{J}^{\tau K}$. By the chain rule

$$\begin{aligned}
\mathbb{P}_q^v\left(\mathcal{E}_{\gamma^*} < \mathcal{J}^{\tau K}\right) &\geq \sum_{\gamma\in\Gamma} \mathbb{P}_q^v\left(\gamma\right) \cdot \mathbb{P}_q^v\left(\mathcal{E}_{\gamma^*} < \mathcal{J}^{\tau K} \mid \gamma\right) \\
&\geq \sum_{\gamma\in\Gamma} \mathbb{P}_q^v\left(\gamma\right) \cdot \mathbb{P}_{q'}^{v'}\left(\mathcal{E}_{\gamma^*} < \mathcal{J}^{K}\right) \\
&\geq \epsilon \cdot \epsilon'.
\end{aligned}$$

□

Proposition 2.8. *If a vector memory random walk is quickly escaping, then for all finite paths $\gamma^* \in \Pi(v_0)$ and all $v \in V$,*

$$\mathbb{P}_q^v(\mathcal{E}_{\gamma^*} < \infty) = 1.$$

Proof. We have by the above corollary $\mathbb{P}_q^v(\mathcal{E}_{\gamma^*} \geq \mathcal{J}^K) \leq 1 - \epsilon$ and for all $n > 0$

$$\mathbb{P}_q^v(\mathcal{E}_{\gamma^*} \geq \mathcal{J}^{K^n}) \leq (1-\epsilon)^n.$$

Letting n tend to ∞, we obtain

$$\mathbb{P}_q^v(\mathcal{E}_{\gamma^*} = \infty) = 0.$$

□

Remark 2.9. *In particular one can define, almost everywhere, a first return map for the win-lose induction to the subsimplex of parameters whose path starts with γ^*. This is what is done in subsection 3.1 where we show that this acceleration of the algorithm is uniformly expanding. This characteristic, in turn, implies the ergodicity of both the acceleration and the initial induction. Such an acceleration serves as the initial step for the dynamical analysis of the algorithm, wherein the distortion will no longer influence the dynamics.*

The subsequent estimate presents a discrete counterpart of an exponential tail property which will play a key role in applying thermodynamic formalism in the subsequent section. It suggests that return times for the mentioned acceleration of the induction can be though of as essentially bounded.

While we state the result for arbitrary distortion, we will only apply it to one specific vector, namely $\mathbf{1} := (1, \ldots, 1)$ at each coordinate. As mentioned in the previous remark, distortion was instrumental in deriving such a structural result but will no longer be relevant in the thermodynamic study.

Corollary 2.10. *If a vector memory random walk is quickly escaping then for any path γ^* there exists $C > 1, \eta > 0$ such that for all $v \in V_0$, $\tau > 1$ and all $q \in \mathbb{R}_+^{\mathcal{A}}$*

$$\mathbb{P}_q^v(\mathcal{J}^\tau \leq \mathcal{E}_{\gamma^*}) < C \cdot \tau^{-\eta}.$$

Proof. Consider K and ϵ as in the previous corollary. Let $\tau > 1$ and $n = \left\lceil \frac{\log \tau}{\log K} \right\rceil$,

$$\mathbb{P}_q^v(\mathcal{J}^\tau \leq \mathcal{E}_{\gamma^*}) \leq \mathbb{P}_q^v(\mathcal{J}^{K^n} \leq \mathcal{E}_{\gamma^*}) \leq (1-\epsilon)^n \leq (1-\epsilon)^{\frac{\log \tau}{\log K} - 1}.$$

Hence, for $C > \frac{1}{1-\epsilon}$ and $\eta = -\frac{\log(1-\epsilon)}{\log K}$, we have

$$\mathbb{P}_q^v(\mathcal{J}^\tau \leq \mathcal{E}_{\gamma^*}) < C \cdot \tau^{-\eta}.$$

□

2.2 Criterion

We start with a simple example of a subgraph in a win-lose induction which prevents a vector memory random walk to be quickly escaping. This serves as a motivation for a criterion on subgraphs that induces quick escape property, as introduced in what follows.

2.2.1 Counter example

Assume a graph defining a win-lose induction admits a vertex v with three outgoing edges as in Figure 5. Where the edge labeled δ points to any vertex in the graph.

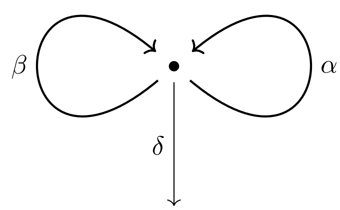

Figure 5: A stable subgraph

Consider the stopping time $\mathscr{L}_\delta$, which corresponds to the property that δ loses at the last step. The following lemma states that if the ratio between q_δ and q_α or q_β is large enough, then the probability that a path leaves the subgraph in finite time is small.

Proposition 2.11. *For all $q \in \mathbb{R}_+^{\mathcal{A}}$*

$$\mathbb{P}_q^v(\mathscr{L}_\delta < \infty) \leq (\phi + \phi^{-1}) \cdot \frac{q_\delta}{\min(q_\alpha, q_\beta)}$$

where ϕ is the golden ratio.

Proof. Let us assume that $q_\beta \leq q_\alpha$. Notice that

$$\mathbb{P}_q^v(\delta \mid \beta \text{ or } \delta) = \frac{q_\delta}{q_\delta + q_\beta} \leq \frac{q_\delta}{q_\beta}.$$

Let $\mathcal{W}_\alpha$ be the property on finite paths that the letter α wins at the last step. As q_δ is unchanged before it loses and q_β is non-decreasing, using the chain rule (5),

$$\mathbb{P}_q^v(\mathscr{L}_\delta < \mathscr{L}_\beta) = \sum_{n=0}^{\infty} \mathbb{P}_q^v(\mathcal{W}_\alpha = n+1) \cdot \mathbb{P}_{qM_\alpha^n}^v(\delta \mid \beta \text{ or } \delta) \leq \frac{q_\delta}{q_\beta}.$$

Let us define $q^{(0)}$ and $q^{(1)}$ to be equal to q and for a sequence of finite paths $\gamma_1, \gamma_2, \ldots$ such that the last step is the only one where β loses, which factors a random infinite path $\overline{\boldsymbol{\gamma}}$. In the induced distortion vector $q^{(2)} = \gamma \cdot q^{(1)}$ we have $q_\beta^{(2)} \geq q_\alpha^{(1)} + q_\beta^{(1)} = q_\alpha^{(1)} + q_\beta^{(0)} \geq q_\alpha^{(1)} = q_\alpha^{(2)}$. Switching α and β, we start again with $q_\alpha^{(2)} \leq q_\beta^{(2)}$ and define, for γ the finite path such that the last step is the only one where α loses, $q^{(3)} = \gamma \cdot q^{(1)}$ where $q_\alpha^{(3)} \geq q_\beta^{(2)} + q_\alpha^{(2)} = q_\beta^{(2)} + q_\alpha^{(1)} \geq q_\beta^{(2)} = q_\beta^{(3)}$.

Let F_n be the Fibonacci sequence such that $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Hence, we construct a sequence of distortion vectors $q^{(n)}$ such that for odd n, $q_\beta \le q_\alpha$ and $\frac{q_\beta^{(n)}}{q_\beta} \ge F_n$ and for even n, $q_\alpha \le q_\beta$ and $\frac{q_\alpha^{(n)}}{q_\beta} \ge F_n$.

Using the chain rule,

$$\mathbb{P}_q^v(\mathscr{L}_\delta < \infty) \le \frac{q_\delta}{q_\beta^{(1)}} + \frac{q_\delta}{q_\alpha^{(2)}} + \frac{q_\delta}{q_\beta^{(3)}} + \cdots \le \frac{q_\delta}{q_\beta} \cdot \sum_{n=1}^{\infty} F_n^{-1}.$$

To compute this sum, notice that for ϕ the golden ratio, $v_1 = (1, \phi)$ and $v_2 = (1, -\phi^{-1})$ are eigenvectors of eigenvalues ϕ and $-\phi^{-1}$ for the matrix associated to the Fibonacci sequence. And $\phi v_1 + \phi^{-1} v_2 = (\phi + \phi^{-1}, \phi + \phi^{-1})$. Hence, for all $n \ge 0$,

$$F_n = \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\phi + \phi^{-1}} \quad \text{and} \quad \sum_{n=1}^{\infty} F_n^{-1} \le (\phi + \phi^{-1}) \cdot \sum_{n=1}^{\infty} \phi^{-n-1} = \phi + \phi^{-1}.$$

□

This proposition implies that when q_δ is sufficiently small compared to q_α and q_β, there is a positive probability that the path in the graph remains indefinitely trapped within the two loops of the subgraph shown in Figure 5.

This stability phenomenon arises from the presence of a subgraph of edges labeled in a subset $\mathcal{L}$ with large distortions which interact with each other. This interaction causes the distortion within $\mathcal{L}$ to increase exponentially, while the distortion of labels outside $\mathcal{L}$ remains constant. As a result, the probability for a label exiting the subgraph to lose in finite time adds up to a value strictly less than 1.

2.2.2 Kerckhoff lemma

We now describe a phenomenon on vector memory random walks first described by Kerckhoff in [Ker85] and enriched in [AGY06] both in the specific setting of Rauzy–Veech induction. We generalize it to the case when the base graph satisfies certain conditions, that we call quick escape property, thus preventing the scenario discussed in the previous subsection from occurring.

Let $\alpha \in \mathcal{A}$, $q \in \mathbb{R}_+^{\mathcal{A}}$, and $\tau > 1$. For a finite path γ, we define the following properties:

- $\mathcal{W}_\alpha$: holds if the letter α wins along the path.
- $\mathcal{J}_{\alpha,q}^\tau$: holds if the distortion vector q *jumps* by a factor of τ along γ, *i.e.*

  $$(q \cdot \gamma)_\alpha \ge \tau \cdot q_\alpha.$$

  When computing a probability $\mathbb{P}_q^v$ we simply write $\mathcal{J}_\alpha^\tau$ when distortion vector is the same.

For $\mathcal{L} \subset \mathcal{A}$, we generalize:

- $\mathcal{W}_\mathcal{L}$: holds if a letter in $\mathcal{L}$ wins along the path.
- $\mathcal{J}_{\mathcal{L},q}^\tau$: holds if

  $$\prod_{\mathcal{L}} (q \cdot \gamma)_\alpha \ge \tau \cdot \prod_{\mathcal{L}} q_\alpha.$$

Under Standing Assumption 2.1 we have the following key lemma.

Lemma 2.12 (Kerckhoff lemma)**.** *For all $\tau > 1$, $q \in \mathbb{R}_+^{\mathcal{A}}$ and all vertex $v \in V_0$, the probability that the vector q jumps by τ on a coordinate in $\mathcal{L}$ before a letter in $\mathcal{L}$ wins satisfies*

$$\mathbb{P}_q^v\left(\mathcal{W}_{\mathcal{L}} \leq \min_{\mathcal{L}} \mathcal{J}_\alpha^\tau\right) > 1 - \frac{1}{\tau^{\sigma|\mathcal{L}|}}.$$

This result indicates that even when a subset of letters has a large distortion, there exists a lower bound on the probability for winning in a reasonable amount of time, implying that it should happen eventually almost surely. When such an event occurs and the winning letter has a small distortion compared to the losing one, it effectively balances the distortion vector. Therefore, by imposing conditions on the graph that favor such subsets of letters with large distortions to win against those with smaller distortions, we can induce the quickly escaping property in the random walk. This forms the core of the next subsection.

Proof of Kerckhoff lemma. We start by describing the argument in the case where $\mathcal{L} = \{\alpha\}$.

let $\mathcal{W}_\alpha^k$ be the property on a finite path γ that the letter α is wins along γ or its length satisfies $|\gamma| \geq k$. We prove by induction on k that for all $k \in \mathbb{N}$ and $\tau > 0$

$$\mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^k\right) \leq \frac{1}{\tau^\sigma}.$$

When $\tau \leq 1$ this property is clearly satisfied, we then assume $\tau > 1$ in the following. For $k = 1$, since the stopping time associated to $\mathcal{W}_\alpha^1$ is bounded by 1 and $\mathcal{J}_\alpha^\tau$ is positive for $\tau > 1$, we have $\mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^1\right) = 0 \leq 1/\tau^\sigma$.

Assume now that the inequality is true for some $k \geq 1$. If α wins along e, the stopping time associated to $\mathcal{W}_\alpha^{k+1}$ is 1 and $\mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < 1\right) = 0$. Hence

$$\mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^{k+1}\right) = \sum_{e \notin \mathcal{W}_\alpha} \mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^{k+1} \mid e\right) \cdot \mathbb{P}_q^v(e)$$

Notice that $e^{-1}\mathcal{J}_{\alpha,q}^\tau = \mathcal{J}_{\alpha,q\cdot e}^{\tau'}$ and if e does not satisfy $\mathcal{W}_\alpha$, $e^{-1}\mathcal{W}_\alpha^{k+1} = \mathcal{W}_\alpha^k$ with $\tau' = \tau \cdot \dfrac{q_\alpha}{(q \cdot e)_\alpha}$. Thus by Proposition 1.10 and relation (6)

$$\mathbb{P}_q^v\left(\mathcal{J}_{\alpha,q}^\tau < \mathcal{W}_\alpha^{k+1} \mid e\right) = \mathbb{P}_{q\cdot e}^{v'}\left(\mathcal{J}_{\alpha,q\cdot e}^{\tau'} < \mathcal{W}_\alpha^k\right).$$

The recurrence hypothesis applied to the constant τ' implies

$$\mathbb{P}_q^v\left(\mathcal{J}_{\alpha,q}^\tau < \mathcal{W}_\alpha^{k+1} \mid e\right) \leq \frac{1}{\tau^\sigma} \cdot \left(\frac{(q \cdot e)_\alpha}{q_\alpha}\right)^\sigma.$$

If there is an edge e going out of v labeled by α, then every other edge must satisfy $\mathcal{W}_\alpha$. By assumptions (K)-**b.** and Proposition 1.9

$$\mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^{k+1}\right) \leq \frac{1}{\tau^\sigma} \cdot \left(\frac{(q \cdot e)_\alpha}{q_\alpha}\right)^\sigma \cdot \mathbb{P}_q^v(e) \leq \frac{1}{\tau^\sigma}.$$

On the contrary, if the label α does not appear in $l(E_v)$ we have, by assumption (K)-**a.**, $(q \cdot e)_\alpha = q_\alpha$ and

$$\mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^{k+1}\right) \leq \sum_{e \notin \mathcal{W}_\alpha} \frac{1}{\tau^\sigma} \cdot \mathbb{P}_q^v(e) = \frac{1}{\tau^\sigma}.$$

Hence

$$\lim_{k\to\infty} \mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha^k\right) = \mathbb{P}_q^v\left(\mathcal{J}_\alpha^\tau < \mathcal{W}_\alpha\right) \leq \frac{1}{\tau^\sigma}.$$

For the general setting, we apply the same reasoning to prove by induction on k that $\mathbb{P}_q^v\left(\mathcal{J}_{\mathcal{L},q}^\tau < \mathcal{W}_{\mathcal{L}}^k\right) \leq \frac{1}{\tau^\sigma}$. The theorem then follows from the fact that $\min_{\mathcal{L}} \mathcal{J}_{\alpha,q}^\tau \leq \mathcal{J}_{\mathcal{L},q}^{\tau|\mathcal{L}|}$. For an edge e going out of v,

$$e^{-1}\left(\mathcal{J}_{\mathcal{L},q}^\tau\right) = \mathcal{J}_{\mathcal{L},q\cdot e}^{\tau'} \quad \text{where} \quad \tau' = \tau \cdot \frac{\prod_{\mathcal{L}} q_\alpha}{\prod_{\mathcal{L}}(q\cdot e)_\alpha}.$$

If e does not satisfy $\mathcal{W}_\alpha$ for any $\alpha \in \mathcal{L}$,

$$e^{-1}\left(\mathcal{W}_{\mathcal{L}}^{k+1}\right) = \mathcal{W}_{\mathcal{L}}^k.$$

The general case then follows, using the same recurrence argument. □

As noticed in Remark 1.11, relation (6) still holds for the probability law conditioned by a prefix invariant set of paths. Thus, the lemma can be generalized for conditional probabilities.

Lemma 2.13 (Conditional Kerckhoff lemma)**.** *Assume $\mathcal{C}$ is a set of paths of non-zero measure such that for all path γ, $\gamma^{-1}\mathcal{C} = \mathcal{C}$ almost surely. For all $\tau > 1$, $q \in \mathbb{R}_+^{\mathcal{A}}$ and all vertex $v \in V_0$, the probability that the vector q jumps by τ on a coordinate in $\mathcal{L}$ before a letter in $\mathcal{L}$ wins satisfies*

$$\mathbb{P}_q^v\left(\mathcal{W}_{\mathcal{L}} \leq \min_{\mathcal{L}} \mathcal{J}_\alpha^\tau \mid \mathcal{C}\right) > 1 - \frac{1}{\tau^{\sigma|\mathcal{L}|}}.$$

2.2.3 A first criterion

The main idea here is to consider degeneration of the induction where for some subsets of labels $\mathcal{L} \subset \mathcal{A}$ the distortion vector at these corresponding coordinates is infinitely larger than for others. In particular, when we are on a vertex that has an outgoing edge labeled in $\mathcal{L}$, any edges with a label outside of $\mathcal{L}$ will almost surely not be chosen.

Let us denote by $G_{\mathcal{L}}$ the subgraph of G, with the same set of vertices V but for which we remove edges along which a letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$. The set of outgoing edges of a vertex $v \in V$ is denoted by $E_v^{\mathcal{L}}$ in $G_{\mathcal{L}}$ and is defined as follows.

- If there is at least one edge in the outgoing edges E_v labeled in $\mathcal{L}$,

$$E_v^{\mathcal{L}} = \{e \in E_v \mid l(e) \in \mathcal{L}\},$$

- Otherwise,

$$E_v^{\mathcal{L}} = E_v.$$

Definition 2.14. *We say that the base graph G of a vector memory random walk is non-degenerating if for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and all v in a strongly connected component $\mathscr{C}$ of $G_{\mathcal{L}}$ one of these properties is true*

1. *There is a path from v leaving $\mathscr{C}$ in which each edge is labeled in $\mathcal{L}$.*

2. $|l(E_v) \cap \mathcal{L}| \leq 1$.

In plain words, this property states that from any vertex, no letter in $\mathcal{L}$ can win against another letter in $\mathcal{L}$ in any strongly connected component of $G_{\mathcal{L}}$ except if there is a path labeled in $\mathcal{L}$ leaving the component.

It is satisfied by a very large class of examples such as the Rauzy–Veech induction and most of multidimensional continued fractions algorithms as showed in [Fou25b] and will be the reason for their ergodicity and several other interesting dynamical properties.

To link our work with previous studies of Rauzy–Veech induction, we include a proof of the fact that Rauzy–Veech induction satisfies this property. This aims to provide some connections for a reader who is familiar with this setting. For more detailed discussions, see [Fou25a].

Proposition 2.15. *The connected component of the Rauzy diagram associated to an irreducible permutation is non-degenerating.*

Proof. In the degenerate subgraph $G_{\mathcal{L}}$, a subset of labels $\mathcal{L}$ always loses against labels in its complementary set $\overline{\mathcal{L}}$. If an interval labeled in $\overline{\mathcal{L}}$ is at the right-hand side extremity (for induction on the right) of the interval exchange it can only lose to a letter in $\overline{\mathcal{L}}$ thus there will always remain an interval labeled in $\overline{\mathcal{L}}$ at the extremity after an arbitrary number of steps. In such a configuration, we cannot have two letters in $\mathcal{L}$ playing with each other.

Moreover, it is well known that on an irreducible Rauzy–Veech induction, all letter must win infinitely many times since the length of the interval must go to zero. Thus if the two extremal intervals are labeled in $\mathcal{L}$ then there is a path labeled in $\mathcal{L}$ to an interval exchange with an extremal interval labeled in $\overline{\mathcal{L}}$. ☐

As an illustration, the reader can check directly these properties on the Rauzy diagram for 3-interval exchange transformations represented on Figure 6 as a win-lose induction.

Notice that our representation of Rauzy diagrams is slightly different from the classical representation where edges are labeled by the words *top* or *bottom* telling which of the top or bottom interval wins whereas we label edges by the corresponding losing letter.

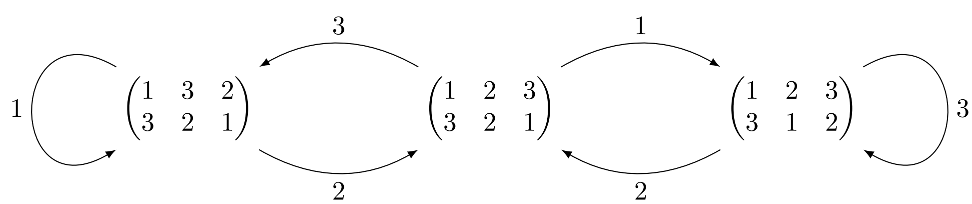

Figure 6: Rauzy diagram for 3-IET.

Remark. *The fully subtractive algorithm in dimension 3 or larger is described in [Fou25b] as the graph with a single vertex and edges looping. It provides a simple example of a win-lose induction that is neither non-degenerating nor quickly escaping nor ergodic. The Poincaré algorithm in dimension 4 is a case that is not non-degenerating but is conjecturally ergodic.*

We now have all the ingredients to show one of our main theorems.

Theorem 2.16. *Assume a vector memory random walk satisfies that there exists $\tau > 1$ such that for all $\alpha \in \mathcal{A}$, every vertex v, and any distortion vector q, we have $\mathbb{P}_q^v(\mathcal{J}_\alpha^\tau < \infty) = 1$. If the base graph is non-degenerating, then the random walk is quickly escaping.*

Let $\mathscr{C}$ be a strongly connected component of $G_{\mathcal{L}}$. We define the property $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}}$ to be true if the path satisfies $\mathcal{S}_{\mathcal{L}}$ (introduced in Section 2.1) or if it goes through an edge that is not in $\mathscr{C}$. In other words, the stopping time corresponds to the state when the value of the distortion on a coordinate in $\mathcal{A} \setminus \mathcal{L}$ reaches (up to factor B) the level of the initial distortion on $\mathcal{L}$ or leaves the strongly connected component $\mathscr{C}$.

Proposition 2.17. *Consider a vector memory random walk satisfying the assumptions of the theorem. Let $\mathscr{C}$ be a strongly connected component of $G_{\mathcal{L}}$ for a subset $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$. For all vertex v in $\mathscr{C}$, there exists $\tau > 1$ such that for any $K > 1$ and q $(\mathcal{L}, K)$-balanced distortion vector*

$$\mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}}^{\mathscr{C}} \leq \mathcal{J}^{\tau^2}) > \left(1 - \frac{1}{\tau^{\sigma|\mathcal{L}|}}\right) \cdot \left(\frac{1}{A^{|\mathscr{C}|} \cdot \tau K}\right)^{|\mathscr{C}|}$$

where A is the constant in hypothesis (H3) and $|\mathscr{C}|$ is the number of vertices in $\mathscr{C}$.

Proof. First, assume v satisfies Item 1 in Definition 2.14. Then there exists a finite path starting at v and leaving $\mathscr{C}$ via edges labeled in $\mathcal{L}$. Removing loops, we can assume it has at most $|\mathscr{C}|$ steps. By Proposition 2.4,

$$\mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}}^{\mathscr{C}} \leq \mathcal{J}^{\tau}) > \left(\frac{1}{A^{|\mathscr{C}|} \cdot K}\right)^{|\mathscr{C}|}.$$

Assume now that $v \in \mathscr{C}'$, where $\mathscr{C}'$ is the subgraph of $\mathscr{C}$ comprising vertices satisfying Item 2 in Definition 2.14 and edges between them. If a finite path satisfies $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}'}$ before $\mathcal{J}^\tau$ but not $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}}$, it implies that it terminates at a vertex within $\mathscr{C}$ but not within $\mathscr{C}'$, thus coming back to the first case considered above. Moreover, as $\mathcal{J}^\tau$ has not occurred yet, the vector remains $(\mathcal{L}, \tau K)$-balanced, and thus the probability of $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}}$ happening before $\mathcal{J}^\tau$ is bounded from below by $\left(\frac{1}{A^{|\mathscr{C}|} \cdot \tau K}\right)^{|\mathscr{C}|}$. Hence, we have

$$\mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}}^{\mathscr{C}} \leq \mathcal{J}^{\tau^2}) > \mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}}^{\mathscr{C}'} \leq \mathcal{J}^{\tau}) \cdot \left(\frac{1}{|\mathcal{A}|^{|\mathscr{C}|} \cdot \tau K}\right)^{|\mathscr{C}|}.$$

According to Lemma 2.12, for all $\tau > 1$

$$\mathbb{P}_q^v\left(\mathcal{W}_{\mathcal{L}} \leq \min_{\alpha \in \mathcal{L}} \mathcal{J}_\alpha^\tau\right) > 1 - \frac{1}{\tau^{\sigma|\mathcal{L}|}}.$$

The proposition then follows from

$$\mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}}^{\mathscr{C}'} \leq \mathcal{J}^{\tau}) \geq \mathbb{P}_q^v\left(\mathcal{W}_{\mathcal{L}} \leq \min_{\alpha \in \mathcal{L}} \mathcal{J}_\alpha^\tau\right).$$

We prove this inequality by showing inclusion of the events. Assume an infinite path $\overline{\boldsymbol{\gamma}}$ satisfies $\mathcal{W}_{\mathcal{L}}$ before $\min_{\alpha \in \mathcal{L}} \mathcal{J}_\alpha^\tau$. If $\min_{\mathcal{L}} \mathcal{J}_\alpha^\tau > \mathcal{J}^\tau$ then a letter in $\overline{\mathcal{L}}$ becomes the largest letter at the time of the jump $\mathcal{J}^\tau$ thus $\mathcal{S}_{\mathcal{L}}$ is satisfied and $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}'} \leq \mathcal{J}^\tau$. If $\min_{\mathcal{L}} \mathcal{J}_\alpha^\tau \leq \mathcal{J}^\tau$, consider the time $m = \mathcal{W}_{\mathcal{L}}(\overline{\boldsymbol{\gamma}})$. The assumption of the theorem on finite jumping times implies that m is almost surely finite. If the path leaves $\mathscr{C}'$

before time $\mathcal{W}_{\mathcal{L}}(\overline{\boldsymbol{\gamma}}) \leq \mathcal{J}^{\tau}$, we are done. Otherwise, at time m, the winning letter in $\mathcal{L}$ must win against a letter in $\overline{\mathcal{L}}$ and $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}'}$ is satisfied, as noticed in Remark 2.5. Thus $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}'} \leq \mathcal{W}_{\mathcal{L}} \leq \mathcal{J}^{\tau}$. □

Remark 2.18. *Notice that, in the proof, we only used Kerckhoff lemma on vertices in $\mathscr{C}'$ not satisfying Item 1 of Definition 2.14 and for $(\mathcal{L}, K)$-balanced distortion vectors. Thus we only need to check inequality in assumption (K)-**b.** for these cases. This refinement will be discussed in Section 2.2.4.*

We can associate to G a directed acyclic graph which vertices are labeled by strongly connected components and for which we draw an edge between two vertices if there is an edge connecting the two strongly connected components in the graph. It is classically called the condensation graph of a directed graph (see for instance section 3.4 of [BM08]). There are *minimal vertices* in this acyclic graph which have no outgoing edges. Vertices in the corresponding strongly connected components, called the *minimal component*, have no edges pointing to another strongly connected component in G.

We conclude the proof of the theorem by induction on the height within the condensation graph of the given strongly connected component, *i.e.* its distance to the minimal vertices.

Proof of Theorem 2.16. Let H be the maximal height in the condensation graph of G. Let $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and $K > 1$. The previous proposition implies that there exists $\tau > 1$ and $\epsilon_0 > 0$ such that for every strongly connected component $\mathscr{C}$ of $G_{\mathcal{L}}$ and every vertex v in $\mathscr{C}$, if q is $(\mathcal{L}, \tau^H K)$-balanced,

$$\mathbb{P}_q^v\left(\mathcal{S}_{\mathcal{L}}^{\mathscr{C}} < \mathcal{J}^{\tau}\right) > \epsilon_0. \tag{8}$$

The inequality can be made strict between the two stopping times simply by increasing τ.

We prove the following property by induction on $h \in \{0, \dots, H\}$:
if v is in a strongly connected component of height h in the condensation graph then for all q $(\mathcal{L}, \tau^{H-h}K)$-balanced distortion vector

$$\mathbb{P}_q^v\left(\mathcal{S}_{\mathcal{L}} < \mathcal{J}^{\tau^{h+1}}\right) > \epsilon_0^{h+1}.$$

Initialization If $h = 0$, v is in a minimal strongly connected component $\mathscr{C}$ for which no edges in $G_{\mathcal{L}}$ are going out of the component thus $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}} = \mathcal{S}_{\mathcal{L}}$ and inequality (8) implies the induction property.

Induction Assume the property is satisfied for $h < H$. Let v be in a component $\mathscr{C}$ of height $h+1$ and q a $(\mathcal{L}, \tau^{H-(h+1)}K)$-balanced — thus $(\mathcal{L}, \tau^H K)$-balanced — distortion vector. By Equation (8), a path $\overline{\boldsymbol{\gamma}}$ satisfies $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}} < \mathcal{J}^{\tau}$ with probability larger than ϵ_0. Let us consider γ the (almost surely) finite prefix of $\overline{\boldsymbol{\gamma}}$ up to time $\mathcal{S}_{\mathcal{L}}$.

If $\mathcal{S}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}}^{\mathscr{C}}$ the path directly satisfies $\mathcal{S}_{\mathcal{L}} < \mathcal{J}^{\tau^{h+2}}$. Otherwise, $\mathcal{S}_{\mathcal{L}} > \mathcal{S}_{\mathcal{L}}^{\mathscr{C}}$ and at time $\mathcal{S}_{\mathcal{L}}^{\mathscr{C}}$ the path must go through an edge in $G_{\mathcal{L}}$ that leaves the component $\mathscr{C}$ to another strongly connected component $\mathscr{C}'$ of height h. In this case, $q' := qM_{\gamma}$ is $(\mathcal{L}, \tau^{H-h}K)$-balanced, $\gamma^{-1}\mathcal{S}_{\mathcal{L},q} = \mathcal{S}_{\mathcal{L},q'}$ and $\gamma^{-1}\mathcal{J}_q^{\tau^{h+2}} \geq \mathcal{J}_{q'}^{\tau^{h+1}}$. Hence, by recurrence hypothesis,

$$\mathbb{P}_q^v\left(\mathcal{S}_{\mathcal{L}} < \mathcal{J}^{\tau^{h+2}} \mid \gamma\right) \geq \mathbb{P}_{q'}^{v'}\left(\mathcal{S}_{\mathcal{L}} < \mathcal{J}^{\tau^{h+1}}\right) > \epsilon_0^{h+1}$$

where $v' = v \cdot \gamma \in \mathscr{C}'$. Using the chain rule (5),

$$\mathbb{P}_q^v \left(\mathcal{S}_{\mathcal{L}} < \mathcal{J}^{\tau^{h+2}} \right) > \epsilon_0^{h+2}.$$

□

Let $\mathcal{C}_{\mathcal{L}}$ be the set of paths such that for all $\tau > 1$ and all $\alpha \in \mathcal{L}$ we have $\mathcal{J}_\alpha^\tau < \infty$. Then for any γ_s finite path in G we have $\gamma_s^{-1}\mathcal{C} = \mathcal{C}$. Thus if $\mathcal{C}_{\mathcal{L}}$ has non-zero measure, the probability law conditioned by $\mathcal{C}_{\mathcal{L}}$ satisfies the conditional Kerckhoff Lemma 2.13 and the same proof applies.

Proposition 2.19. *Let $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ such that the set $\mathcal{C}_{\mathcal{L}}$ has non-zero measure. For all $\emptyset \subsetneq \mathcal{L}' \subseteq \mathcal{L}$ and all $K > 1$, there exist $\tau > 1$ and $\epsilon > 0$ such that for all vertex $v \in V$ and all $(\mathcal{L}', K)$-balanced distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$*

$$\mathbb{P}_q^v(\mathcal{S}_{\mathcal{L}'} < \mathcal{J}^\tau \mid \mathcal{C}_{\mathcal{L}}) > \epsilon.$$

Corollary 2.20. *If the base graph of a vector memory random walk is non-degenerating then every vertex v, and any distortion vector q, we have $\mathbb{P}_q^v(\mathcal{C}_{\mathcal{A}}) = 1$. Thus the random walk is quickly escaping.*

Proof. Let $\mathcal{L}$ be as in the proposition and $\mathcal{M}_{\mathcal{L}'}^{\kappa}$ denote (changing $\mathcal{A}$ to $\mathcal{L}$ in the previous definition) the property of a finite path γ for which

$$\kappa \cdot \min_{\mathcal{L}'}\{q \cdot \gamma\} \geq \max_{\mathcal{L}} q.$$

Using the same argument as in Lemma 2.6, we show by induction that for each $n \leq |\mathcal{L}|$, there exist constants $\tau_n > 1$, $\kappa_n > 0$, and $\epsilon_n > 0$ such that for every vertex v and every distortion vector q, there exists a subset $\mathcal{L}' \subset \mathcal{L}$ with $|\mathcal{L}'| = n$ that satisfies:

$$\mathbb{P}_q^v(\mathcal{M}_{\mathcal{L}'}^{\kappa_n} < \mathcal{J}^{\tau_n} \mid \mathcal{C}_{\mathcal{L}}) > \epsilon_n.$$

With an extra step of induction, there exists $\beta \in \mathcal{A} \setminus \mathcal{L}$, $\tau > 1$, $\kappa > 0$, and $\epsilon > 0$ such that

$$\mathbb{P}_q^v(\mathcal{M}_{\mathcal{L}\cup\{\beta\}}^{\kappa} < \mathcal{J}^\tau \mid \mathcal{C}_{\mathcal{L}}) > \epsilon.$$

This implies that there exists $\beta \in \mathcal{A} \setminus \mathcal{L}$ such that $\mathbb{P}_q^v(\mathcal{C}_{\mathcal{L}\cup\{\beta\}} \mid \mathcal{C}_{\mathcal{L}}) = 1$. Thus, using the chain rule, for all $\alpha \in \mathcal{A}$ such that $\mathcal{C}_{\{\alpha\}}$ has non-zero Lebesgue measure, $\mathbb{P}_q^v(\mathcal{C}_{\mathcal{A}} \mid \mathcal{C}_{\{\alpha\}}) = 1$ and finally $\mathbb{P}_q^v(\mathcal{C}_{\mathcal{A}}) = \mathbb{P}_q^v\left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{C}_{\{\alpha\}}\right) = 1$ where the last equality follows from Remark 2.2. □

Remark 2.21. *The non-degenerating assumption implies that the graph cannot be labeled by a strict subset $\mathcal{L}$ of $\mathcal{A}$. Indeed, if we consider a minimal connected component, the only way it satisfies Item 2 of non-degenerating property would be to have only one outgoing edge at each vertex contradicting the Standing Assumptions 2.1.*

2.2.4 Case of vertices with a unique outgoing edge

In the following, we try to weaken the second assumption together with (K) of Standing Assumptions 2.1.

Assume there exists vertices with a unique outgoing edges in the base graph G. As mentioned in Section 1.2.2, one prefer to consider the accelerated random walk which continues as long as the end vertex has a unique outgoing edge: Nevertheless, such accelerated random walks do not usually satisfy hypothesis (K) even when G

does. We show that Properties (K)-**a.** and (K)-**b.** on such graphs can be relaxed to Properties (P_2) and (P_3) in the following generalized non-degenerating graph definition. These weaker properties still imply quick escape property and have the advantage that they can be checked as a graph property for induced random walks on subgraph of G which satisfies (K).

We introduce a generalized definition of the non-degenerating property. Recall that we say $v \in V$ is a *branching vertex* if it has two or more outgoing edges. A path which start and end vertices, and only them two, belong to $\widehat{V}$ is called a *branch path*. The label of such a path refers to the label of its first edge.

Definition 2.22. *The base graph G of a vector memory random walk is called non-degenerating if, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and every branch path γ contained in a strongly connected component $\mathscr{C}$ of $G_{\mathcal{L}}$, one of these properties is true.*

(P_1) *There exists a path in G starting at a vertex of γ which leaves $\mathscr{C}$ and such that each edge based at a branching vertex in the path is labeled in $\mathcal{L}$.*

(P_2) *The branch path γ acts trivially on coordinates in $\mathcal{L}$ of a distortion vector.*

(P_3) *There are no other branch path than γ starting from the same vertex v and contained in $G_{\mathcal{L}}$. Moreover, there exists $M > 1$ such that for all $\alpha \in \mathcal{L}$ and all $(\mathcal{L}, K)$-balanced distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$*

$$(q \cdot \gamma)_\alpha \leq q_\alpha + M \cdot \sum_{\substack{\beta \in l(E_v) \\ \beta \neq l(\gamma)}} q_\beta .$$

As for Property (K)-**b.**, Property (P_3) implies a bound on probability of the corresponding branch path by its increase in distortion.

Proposition 2.23. *If a branch path γ satisfies Property (P_3) then for all $K > 1$, there exists $\sigma > 0$ such that for all $(\mathcal{L}, K)$-balanced distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$,*

$$\mathbb{P}_q^v(\gamma) \leq \left(\frac{\prod_{\mathcal{L}} q_\alpha}{\prod_{\mathcal{L}} (q \cdot \gamma)_\alpha} \right)^\sigma .$$

Proof. Let α be the label of γ and $Q = \sum_{\beta \in l(E_v)} q_\beta$. Let us prove that there exists $\sigma > 0$ such that for all $\delta \in \mathcal{L}$,

$$\frac{q_\alpha}{Q} \leq \left(\frac{q_\delta}{(q \cdot \gamma)_\delta} \right)^\sigma .$$

The proof is similar to the one of Proposition 1.9. As q is $(\mathcal{L}, K)$-balanced and δ belongs to $\mathcal{L}$, $q_\alpha \leq K q_\delta$, thus

$$\frac{q_\alpha}{Q} = \frac{1}{1 + \dfrac{Q - q_\alpha}{q_\alpha}} \leq \frac{q_\delta}{q_\delta + K^{-1}(Q - q_\alpha)} .$$

Moreover, by assumption (P_3), $\dfrac{q_\delta}{q_\delta + M(Q - q_\alpha)} \leq \dfrac{q_\delta}{(q \cdot \gamma)_\delta}$ then one is looking for $\sigma > 0$ such that

$$\frac{q_\delta}{q_\delta + K^{-1}(Q - q_\alpha)} \leq \left(\frac{q_\delta}{q_\delta + M(Q - q_\alpha)} \right)^\sigma \iff \log(1 + K^{-1} \frac{Q - q_\alpha}{q_\delta}) \geq \sigma \log \left(1 + M \cdot \frac{Q - q_\alpha}{q_\delta} \right) .$$

As $M \cdot \frac{Q-q_\alpha}{q_\delta} \geq \log(1 + M \cdot \frac{Q-q_\alpha}{q_\delta})$, one only has to find σ such that

$$\log\left(1 + K^{-1}\frac{Q - q_\alpha}{q_\delta}\right) \geq \sigma M \cdot \frac{Q - q_\alpha}{q_\delta}.$$

As $\dfrac{Q - q_\alpha}{q_\delta} \leq K \cdot |\mathcal{A}|$, by concavity of $\log(1+x)$,

$$\log\left(1 + K^{-1} \cdot \frac{Q - q_\alpha}{q_\delta}\right) \geq \log(1 + |\mathcal{A}|) \cdot \frac{1}{|\mathcal{A}|} \cdot K^{-1} \cdot \frac{Q - q_\alpha}{q_\delta}$$

thus one only has to pick $\sigma \leq \frac{\log(1+|\mathcal{A}|)}{M \cdot K \cdot |\mathcal{A}|}$. We conclude by observing that

$$\mathbb{P}_q^v(\gamma) = \frac{q_\alpha}{Q} \leq \left(\frac{\prod_{\mathcal{L}} q_\delta}{\prod_{\mathcal{L}} (q \cdot \gamma)_\delta}\right)^{\frac{\sigma}{|\mathcal{A}|}}.$$

□

Any finite path in G from and to a branching vertex can be decomposed as a unique sequence of branch paths $\gamma_1 \ldots \gamma_n$. We define stopping times at branching vertices for index n, characterized by the following properties :

- $\mathcal{W}_\alpha$, $\mathcal{W}_\mathcal{L}$: holds if the letter α or a letter in $\mathcal{L}$ wins along the path.
- $\mathcal{J}^\tau_{\alpha,q}$: holds if the distortion vector q *jumps* by a factor of τ along $\gamma_1 \ldots \gamma_n$, *i.e.*

$$(q \cdot \gamma_1 \ldots \gamma_n)_\alpha \geq \tau \cdot q_\alpha.$$

Let $\mathscr{C}$ be a strongly connected component in $G_\mathcal{L}$, and $\mathscr{C}'$ be the subcomponent in which vertices satisfying (P_1) are removed (as well as the edges pointing to them). We define a stopping time associated to the property $\mathcal{W}_\alpha^{\mathscr{C}'}$ which holds if $\mathcal{W}_\alpha$ is satisfied or the path visits an edge outside of $\mathscr{C}'$.

The same results, namely Theorem 2.16 and Corollary 2.20, then follow from a refined version of Kerckhoff lemma.

Lemma 2.24 (Refined Kerckhoff lemma)**.** *Let v be a vertex in $\mathscr{C}'$. For all $\tau > 1$ and $q \in \mathbb{R}_+^\mathcal{A}$, the probability that the vector q jumps by τ on a coordinate in $\mathcal{L}$, before a letter in $\mathcal{L}$ wins or the path leaves $\mathscr{C}'$, satisfies*

$$\mathbb{P}_q^v\left(\mathcal{W}_\mathcal{L}^{\mathscr{C}'} \leq \min_\mathcal{L} \mathcal{J}_\alpha^\tau\right) > 1 - \frac{1}{\tau^{\sigma|\mathcal{L}|}}.$$

Proof. As in Lemma 2.12, let $(\mathcal{W}_\mathcal{L}^{\mathscr{C}'})^k$ be the property for a path $\gamma = \gamma_1 \ldots \gamma_n$ to satisfy $\mathcal{W}_\mathcal{L}^{\mathscr{C}'}$ or $n \geq k$. We prove similarly by induction that

$$\mathbb{P}_q^v\left(\mathcal{J}_{\mathcal{L},q}^\tau < \left(\mathcal{W}_\mathcal{L}^{\mathscr{C}'}\right)^k\right) \leq \frac{1}{\tau^{\sigma|\mathcal{L}|}}.$$

□

A graph criterion This non-degenerating property can be checked by a graph property for well chosen subgraphs of a vector memory random walk satisfying assumption (K). For clarity in the following, we denote the graph as $G = (V, E)$ and subgraph $F = (V^F, E^F)$.

Consider a vector memory random walk that satisfies assumption (K) and such that its base graph G has no vertex with less than two outgoing edges. Let F be a subgraph of G such that all its removed edges start from a vertex that has a unique outgoing edge in F.

Definition 2.25. *A subgraph F is called* admissible *if for any $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, every branch path γ contained in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$, starting at a vertex v, satisfies one of the following properties.*

(P_1) *There exists a path in F starting at a vertex of γ which leaves $\mathscr{C}$ and such that each edge based at a branching vertex of F is labeled in $\mathcal{L}$.*

(P_2^K) *All edges in γ are labeled within G by letters not in $\mathcal{L}$.*

(P_3^K) *The branch path γ is labeled by $\lambda \in \mathcal{L}$ and $\mathcal{L} \cap l(E_v^F) = \{\lambda\}$. Moreover, for every edge of G in γ labeled in $\mathcal{L}$, the removed edges in F going out of the starting vertex for this edge are labeled in $l(E_v^F) \setminus \{\lambda\}$.*

Proposition 2.26. *Properties (P_2^K) and (P_3^K) imply (P_2) and (P_3) respectively. In other words, an admissible subgraph is non-degenerating.*

2.2.5 Finer factorizations

The previous weaker condition has introduced the idea of integrating hypothesis (K) into a graph condition. However, for applications, another weaker criterion is needed on factorizations generalizing the acceleration to branching vertices. This section requires additional notation and may feel more technical, but all the key ideas and arguments have already been developed in the preceding sections. Hopefully, by this point, they will have become intuitive to the reader.

Definition 2.27. *Consider, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, a subset $\widetilde{V}^{\mathcal{L}}$ of vertices of $G_{\mathcal{L}}$ such that every loop in this subgraph contains a vertex in $\widetilde{V}^{\mathcal{L}}$.*

Let $\widetilde{E}^{\mathcal{L}}$ be the set of finite paths in $G_{\mathcal{L}}$ for which the start and end vertex belong to $\widetilde{V}^{\mathcal{L}}$ and no other visited vertices do. Paths in $\widetilde{E}^{\mathcal{L}}$ are called $\mathcal{L}$-factor paths.

We say the collection $\{\widetilde{V}^{\mathcal{L}}\}$ is a filling factoring family *if every $\mathcal{L}$-factor path γ visits at most one branching vertex v satisfying $l(E_v) \cap \mathcal{L} \neq \emptyset$ which is not the end vertex of γ.*

Remark. *Notice that the branch path in the previous subsection correspond to factor paths where $\widetilde{V}^{\mathcal{L}}$ is the set of branching vertices for all $\mathcal{L}$.*

A $\mathcal{L}$-factor path of a filling factoring family can be uniquely decomposed as $\gamma = \gamma_s \cdot \gamma_t$, where γ_s ends at this unique visited branching vertex v, when it exists, or is empty if there is no such vertex.

Definition 2.28. *A filling factoring family $\{\widetilde{V}^{\mathcal{L}}\}$ with $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ is a* non-degenerating factorization *of G if there exists $M, C > 1$ such that every $\mathcal{L}$-factor path $\gamma = \gamma_s \cdot \gamma_t$ contained in a strongly connected component $\mathscr{C}$ of $G_{\mathcal{L}}$ satisfies one of (P_1), (P_2), or:*

($\widetilde{P}_3$) *There is a decomposition* $\mathcal{L} = \mathcal{L}_1 \sqcup \mathcal{L}_2$ *such that if* γ *is going through a branching vertex* v*, then it is the unique* $\mathcal{L}$*-factor path going through* v *and the label of the edge in the path based at* v *is contained in* $\mathcal{L}_1$*. Moreover, for all* $(\mathcal{L}, K)$*-balanced distortion vector* $q \in \mathbb{R}_+^{\mathcal{A}}$

$$(q \cdot \gamma)_\alpha \leq q_\alpha + M \cdot \sum_{\substack{\beta \in l(E_v) \\ \beta \neq l(\gamma)}} (q \cdot \gamma_s)_\beta. \quad \textit{for all } \alpha \in \mathcal{L}_1 \tag{9}$$

and

$$\sum_{\alpha \in \mathcal{L}_2} (q \cdot \gamma)_\alpha - q_\alpha \leq C \cdot \sum_{\beta \notin \mathcal{L}} (q \cdot \gamma)_\beta - q_\beta. \tag{10}$$

These two equations give different counter parts for the growth of the distortion on $\mathcal{L}$. On one hand, the increase on a coordinate in $\mathcal{L}_1$ is compensated by the probability that this letter wins, as used before in Kerckhoff lemma. On the other hand, an increase on $\mathcal{L}_2$ implies a comparable increase on coordinates in the complementary set of $\mathcal{L}$ approaching the balancing event $\mathcal{S}_\mathcal{L}$. This will be the heart of the Kerckhoff lemma in this refined setting.

Let us define for $\delta > 0$ a property $\mathcal{D}^\delta_{\mathcal{L},q}$ which is satisfied when

$$\sum_{\beta \notin \mathcal{L}} (q \cdot \gamma)_\beta - q_\beta \geq \log \delta \cdot \min_{\alpha \in \mathcal{L}} q_\alpha.$$

Proposition 2.29. *For* γ *as in* ($\widetilde{P}_3$) *we have for* $\sigma \leq \frac{1}{C \cdot |\mathcal{A}|}$

$$\gamma^{-1} \mathcal{D}^\delta_{\mathcal{L},q} \leq \mathcal{D}^{\delta'}_{\mathcal{L},q\cdot\gamma} \quad \textit{with} \quad \delta' = \delta \cdot \left(\frac{\prod_{\mathcal{L}_2} q_\alpha}{\prod_{\mathcal{L}_2} (q \cdot \gamma)_\alpha} \right)^\sigma.$$

Proof. Let $\rho \geq 0$ such that $e^\rho = \dfrac{\prod_{\mathcal{L}_2} (q \cdot \gamma)_\alpha}{\prod_{\mathcal{L}_2} q_\alpha}$.

There exists $\alpha \in \mathcal{L}_2$ such that $(q \cdot \gamma)_\alpha \geq e^{\frac{\rho}{|\mathcal{A}|}} \cdot q_\alpha$, and by (10),

$$\sum_{\beta \notin \mathcal{L}} (q \cdot \gamma)_\beta - q_\beta \geq \frac{1}{C} \cdot (e^{\frac{\rho}{|\mathcal{A}|}} - 1) \cdot q_\alpha \geq \frac{\rho}{C \cdot |\mathcal{A}|} \cdot \min_{\alpha \in \mathcal{L}} q_\alpha.$$

If γ' satisfies $\mathcal{D}^{\delta'}_{\mathcal{L},q\cdot\gamma}$,

$$\sum_{\beta \notin \mathcal{L}} (q \cdot \gamma \cdot \gamma')_\beta - (q \cdot \gamma)_\beta \geq \log \delta' \cdot \min_{\alpha \in \mathcal{L}} (q \cdot \gamma)_\alpha \geq \log \delta' \cdot \min_{\alpha \in \mathcal{L}} q_\alpha$$

and

$$\sum_{\beta \notin \mathcal{L}} (q \cdot \gamma \cdot \gamma')_\beta - q_\beta \geq \log \left(\delta' \cdot e^{\frac{\rho}{C \cdot |\mathcal{A}|}} \right) \cdot \min_{\alpha \in \mathcal{L}} q_\alpha$$

thus $\gamma \cdot \gamma'$ satisfies $\mathcal{D}^\delta_{\mathcal{L},q}$. □

We show a bound similar to Proposition 2.23.

Proposition 2.30. *If a* $\mathcal{L}$*-factor path* $\gamma = \gamma_s \cdot \gamma_t$ *satisfies Property* ($\widetilde{P}_3$) *in the latter definition, we have for some* $\sigma > 0$ *and all* $(\mathcal{L}, K)$*-balanced distortion vector* $q \in \mathbb{R}_+^{\mathcal{A}}$,

$$\mathbb{P}^v_q(\gamma) \leq \left(\frac{\prod_{\mathcal{L}_1} q_\alpha}{\prod_{\mathcal{L}_1} (q \cdot \gamma)_\alpha} \right)^\sigma \cdot \mathbb{P}^v_q(\gamma_s).$$

Proof. Let α be the label of the first edge of γ_t. The probability of going through γ is determined by the choice at its branching vertex

$$\mathbb{P}_q^v(\gamma) = \frac{(q \cdot \gamma_s)_\alpha}{\sum\limits_{\beta \in l(E_v)} (q \cdot \gamma_s)_\beta} \cdot \mathbb{P}_q^v(\gamma_s).$$

Observe that by (H3) and the fact that q is $(\mathcal{L}, K)$-balanced, assuming $M \geq A^{|V|}$, there exists $0 \leq a \leq M \cdot q_\alpha$ such that

$$\frac{(q \cdot \gamma_s)_\alpha}{\sum\limits_{\beta \in l(E_v)} (q \cdot \gamma_s)_\beta} = \frac{q_\alpha + a}{q_\alpha + \sum\limits_{\substack{\beta \in l(E_v) \\ \beta \neq \alpha}} (q \cdot \gamma_s)_\beta + a}$$

Let us denote, for readability, $q = q_\alpha$ and $Q = q_\alpha + \sum\limits_{\substack{\beta \in l(E_v) \\ \beta \neq \alpha}} (q \cdot \gamma_s)_\beta$. We show that for all $a \leq M \cdot q$ and $q \leq Q$, there exists $\sigma_1 > 0$ such that

$$\frac{q + a}{Q + a} \leq \left(\frac{q}{Q} \right)^{\sigma_1}.$$

First, notice that $\frac{q+a}{Q+a} \leq \frac{q+M \cdot q}{Q+M \cdot q}$ since the expression is increasing with respect to a. Now,

$$\frac{q + M \cdot q}{Q + M \cdot q} \leq \left(\frac{q}{Q} \right)^{\sigma_1} \iff (1 + M) \cdot \left(\frac{q}{Q} \right)^{1-\sigma_1} \leq 1 + M \cdot \frac{q}{Q}.$$

A study of the function $(1+M) \cdot x^{1-\sigma_1} - M \cdot x$ for $x \in [0,1]$ shows that this inequality is true for $\sigma_1 \leq \frac{1}{1+M}$.

So we have $\mathbb{P}_q^v(\gamma) \leq \left(\frac{q_\alpha}{Q} \right)^{\sigma_1} \cdot \mathbb{P}_q^v(\gamma_s)$. By the same argument as in Proposition 2.23, we have $\sigma_2 > 0$ such that for all $\delta \in \mathcal{L}_1$,

$$\frac{q_\alpha}{Q} \leq \left(\frac{q_\delta}{(q \cdot \gamma)_\delta} \right)^{\sigma_2}.$$

We then conclude taking $\sigma = \sigma_1 \cdot \sigma_2 / |\mathcal{L}_1|$. ☐

Proposition 2.31. *A refined Kerckhoff lemma on the factorized paths holds and so does its consequence Proposition 2.17 as well as Theorem 2.16 and Corollary 2.20 derived from it.*

Proof. To consider the two phenomenon of $(\widetilde{P}_3)$ jointly, we define for $\delta > 0$, the stopping time $\mathcal{H}_{\mathcal{L}}^\delta$ as the minimum between $\mathcal{D}_{\mathcal{L},q}^\delta$ and $\mathcal{W}_{\mathcal{L}}^{\mathscr{C}'}$. Here again, let $(\mathcal{H}_{\mathcal{L}}^\delta)^k$ be the property for a path $\gamma = \gamma_1 \ldots \gamma_n$ to satisfy $\mathcal{H}_{\mathcal{L}}^\delta$ or $n \geq k$.

We show a Kerckhoff lemma refined with this new compensating phenomenon. For all $\tau > 0$, $\delta > 0$ and $(\mathcal{L}, K)$-balanced distortion vector $q \in \mathbb{R}_+^{\mathcal{A}}$

$$\mathbb{P}_q^v \left(\mathcal{J}_{\mathcal{L}}^\tau < \mathcal{H}_{\mathcal{L}}^\delta \right) \leq \frac{\delta}{\tau^\sigma}.$$

First notice that case when $\tau \leq 1$ is clear, since in this case $\mathcal{J}_{\mathcal{L}}^\tau = 0$ and either $\delta < 1$ and $\mathcal{H}_{\mathcal{L}}^\delta = \mathcal{J}_{\mathcal{L}}^\tau = 0$ or $\delta \geq 1$ and $\frac{\delta}{\tau^\sigma} \geq 1$. In the following we assume that $\tau > 1$.

Initialization for $k = 1$ is clear since for $\tau > 1$, $\mathbb{P}_q^v \left(\mathcal{J}_{\mathcal{L}}^\tau < (\mathcal{H}_{\mathcal{L}}^\delta)^1 \right) = 0$.

To show induction, let us decompose $\widetilde{E}_v^{\mathcal{L}} = (\widetilde{E}_v^{\mathcal{L}})^{(P_1)} \sqcup (\widetilde{E}_v^{\mathcal{L}})^{(P_2)} \sqcup (\widetilde{E}_v^{\mathcal{L}})^{(\widetilde{P}_3)}$ where each of these subsets is respectively the subset of paths which satisfy

- Properties (P_1)
- Does not satisfy Properties (P_1) but satisfy Property (P_2)
- Does not satisfy Properties (P_1) and (P_2) but satisfy Property $(\widetilde{P}_3)$.

We also define the set of prefixes in the decomposition defined in Proposition 2.30

$$(\widetilde{E}_v^{\mathcal{L}})_s^{(\widetilde{P}_3)} = \left\{ \gamma_s \mid \gamma \in (\widetilde{E}_v^{\mathcal{L}})^{(\widetilde{P}_3)} \text{ and } \gamma = \gamma_s \gamma_t \right\}.$$

Notice that no path in $(\widetilde{E}_v^{\mathcal{L}})_s^{(\widetilde{P}_3)}$ is prefix to another one in the same set as well as not a prefix of a path in $(\widetilde{E}_v^{\mathcal{L}})^{(P_2)}$. Moreover, for all $\gamma_s \in (\widetilde{E}_v^{\mathcal{L}})_s^{(\widetilde{P}_3)}$, there is only one $\mathcal{L}$-factor path $\gamma \in (\widetilde{E}_v^{\mathcal{L}})^{(\widetilde{P}_3)}$ such that γ_s is a prefix of γ. If a path is in $(\widetilde{E}_v^{\mathcal{L}})^{(P_1)}$ then it must leave $\mathscr{C}'$ and satisfies in particular $\mathcal{H}_{\mathcal{L}}^{\delta}$. Applying the chain rule and splitting cases, together with Proposition 2.30 for the second case, we get

$$\mathbb{P}_q^v \left(\mathcal{J}_{\mathcal{L}}^{\tau} \leq (\mathcal{H}_{\mathcal{L}}^{\delta})^{k+1} \right) \leq \sum_{\gamma \in (\widetilde{E}_v^{\mathcal{L}})^{(P_2)}} \mathbb{P}_q^v(\gamma) \cdot \mathbb{P}_{q\cdot\gamma}^{\gamma\cdot v} \left(\mathcal{J}_{\mathcal{L}}^{\tau} < (\mathcal{H}_{\mathcal{L}}^{\delta})^k \right)$$
$$+ \sum_{\gamma_s \in (\widetilde{E}_v^{\mathcal{L}})_s^{(\widetilde{P}_3)}} \left(\frac{\prod_{\mathcal{L}_1} q_\alpha}{\prod_{\mathcal{L}_1} (q\cdot\gamma)_\alpha} \right)^{\sigma} \cdot \mathbb{P}_q^v(\gamma_s) \cdot \mathbb{P}_{q\cdot\gamma}^{\gamma\cdot v} \left(\mathcal{J}_{\mathcal{L}}^{\tau'} < (\gamma^{-1} \mathcal{H}_{\mathcal{L}}^{\delta})^k \right)$$

where

$$\tau' = \tau \cdot \frac{\prod_{\mathcal{L}} q_\alpha}{\prod_{\mathcal{L}} (q\cdot\gamma)_\alpha}.$$

By Proposition 2.29, $\mathbb{P}_{q\cdot\gamma}^{\gamma\cdot v} \left(\mathcal{J}_{\mathcal{L}}^{\tau'} < (\gamma^{-1} \mathcal{H}_{\mathcal{L}}^{\delta})^k \right) \leq \mathbb{P}_{q\cdot\gamma}^{\gamma\cdot v} \left(\mathcal{J}_{\mathcal{L}}^{\tau'} < (\mathcal{H}_{\mathcal{L}}^{\delta'})^k \right)$ with

$$\delta' = \delta \cdot \left(\frac{\prod_{\mathcal{L}_2} q_\alpha}{\prod_{\mathcal{L}_2} (q\cdot\gamma)_\alpha} \right)^{\sigma}.$$

Which is bounded, by recurrence hypothesis, by

$$\frac{\delta'}{(\tau')^{\sigma}} = \frac{\delta}{\tau^{\sigma}} \cdot \left(\frac{\prod_{\mathcal{L}_1} q_\alpha}{\prod_{\mathcal{L}_1} (q\cdot\gamma)_\alpha} \right)^{-\sigma}.$$

Hence

$$\mathbb{P}_q^v \left(\mathcal{J}_{\mathcal{L}}^{\tau} < (\mathcal{H}_{\mathcal{L}}^{\delta})^{k+1} \right) \leq \left(\sum_{\gamma \in (\widetilde{E}_v^{\mathcal{L}})^{(P_2)}} \mathbb{P}_q^v(\gamma) + \sum_{\gamma_s \in (\widetilde{E}_v^{\mathcal{L}})_s^{(\widetilde{P}_3)}} \mathbb{P}_q^v(\gamma_s) \right) \frac{\delta}{\tau^{\sigma}} \leq \frac{\delta}{\tau^{\sigma}}.$$

Notice that $\mathcal{D}_{\mathcal{L}}^{\delta}$ implies $\mathcal{S}_{\mathcal{L}}$ when $\log\delta \geq \frac{2}{B}$. Thus taking such δ, $\mathcal{H}_{\mathcal{L}}$ implies $\mathcal{S}_{\mathcal{L}}$. The remaining of the results are then proved by the same arguments as in previous sections. □

A graph criterion Here again, these properties can be checked with graph properties on a subgraph F of the base graph G of a vector memory random walk satisfying (K).

Consider a vector memory random walk that satisfies assumption (K) and such that its base graph G has no vertex with less than two outgoing edges. Let F be a subgraph of G such that all its removed edges start from a vertex that has a unique outgoing edge in F.

Let us denote the set of winning letters against a label $\alpha \in \mathcal{A}$ along a path γ by

$$W_\gamma^\alpha := \{\beta \in \mathcal{A} \mid \beta \text{ wins against } \alpha \text{ along } \gamma\}.$$

This set can be constructed iteratively. Denote by e_i the i-th edge of γ which starts at vertex v_{i-1}. For $1 \le n \le |\gamma|$, we define $W_\gamma^\alpha(0,n) := \{\alpha\}$ and for $k \ge 1$

$$W_\gamma^\alpha(k,n) := \left\{\beta \in \mathcal{A} \mid \exists 1 \le i \le n,\ \exists \delta \in W_\gamma^\alpha(k-1,i) \text{ such that } l(e_i) = \delta \text{ and } \beta \in E_{v_{i-1}} \setminus \{\delta\}\right\}.$$

Definition 1.7 implies a transitivity property: if a letter β wins against α along γ_1 and δ wins against β along γ_2 then δ wins against α along $\gamma_1 \cdot \gamma_2$. In particular, if $\alpha \in E_{\gamma \cdot v}$, for all $\beta \in E_{\gamma \cdot v} \setminus \{\alpha\}$, $W_\gamma^\beta \subset W_{\gamma \cdot \alpha}^\alpha$.

Moreover, by Remark 1.8 we have under assumption (K),

$$W_\gamma^\alpha = \bigcup_{k=1}^{|\gamma|} W_\gamma^\alpha(k, |\gamma|).$$

Proposition 2.32. *For all path γ in G and label $\alpha \in \mathcal{A}$,*

$$(q \cdot \gamma)_\alpha \le q_\alpha + (M \cdot |\gamma| \cdot |\mathcal{A}|)^{|\gamma|} \cdot \sum_{\beta \in W_\gamma^\alpha} q_\beta.$$

Proof. Let γ_n be the prefix of γ of length n. Assume the proposition is true for γ_{n-1}. Let v be the ending vertex of γ_{n-1} and α be the label of the next edge in γ. On other labels, (K) implies that the distortion is unchanged. Now, by (K) and non-decreasing hypothesis (H1),

$$\begin{aligned}
(q \cdot \gamma_n)_\alpha &\le q_\alpha + M \cdot n \cdot \sum_{\beta \in l(E_v) \setminus \{\alpha\}} (q \cdot \gamma_{n-1})_\beta \\
&\le q_\alpha + M \cdot n \cdot \left(\sum_{\beta \in l(E_v) \setminus \{\alpha\}} q_\beta + M^{n-1} \cdot (n-1)^{n-1} \cdot |\mathcal{A}|^{n-1} \cdot \sum_{\delta \in W_{\gamma_{n-1}}^\beta} q_\beta \right) \\
&\le q_\alpha + M \cdot n \cdot \left(1 + M^{n-1} \cdot (n-1)^{n-1} \cdot |\mathcal{A}|^n\right) \cdot \sum_{\beta \in W_{\gamma_n}^\alpha} q_\beta \\
&\le q_\alpha + (M \cdot n \cdot |\mathcal{A}|)^n \cdot \sum_{\beta \in W_{\gamma_n}^\alpha} q_\beta
\end{aligned}$$

□

We introduce in this context the following graph property on a $\mathcal{L}$-factor path $\gamma = \gamma_s \cdot \gamma_t$.

($\widetilde{P}_3^K$) γ is going through a branching vertex v and there is a splitting $\mathcal{L} = \mathcal{L}_1 \sqcup \mathcal{L}_2$ such that if and λ is the label of the first edge of γ_t then $l(E_v^F) \cap \mathcal{L} = \{\lambda\}$ and $\lambda \in \mathcal{L}_1$. Moreover,

- Each label winning against a letter in $\mathcal{L}_1$ along γ is either in $l(E_v^F) \setminus \{\lambda\}$ or wins against a letter of this set along γ_s.
- For each label β winning along an edge γ_j labeled in $\mathcal{L}_2$, there exists an index i such that β wins along the edge γ_i which is not labeled in $\mathcal{L}$. Moreover, if $i < j$, the label β does not lose along the subpath $\gamma_i \dots \gamma_j$.

Proposition 2.33. $(\widetilde{P}_3^K)$ *implies* $(\widetilde{P}_3)$.

Proof. By Proposition 2.32, there is a constant M_1 such that for labels in $\mathcal{L}_1$

$$(q \cdot \gamma)_\alpha \leq q_\alpha + M_1 \cdot \sum_{\beta \in W_\gamma^\alpha} q_\beta.$$

The first point of condition $(\widetilde{P}_3^K)$ implies that

$$\sum_{\beta \in W_\gamma^\alpha} q_\beta \leq B \cdot \sum_{\substack{\beta \in l(E_v^F) \\ \beta \neq \lambda}} (q \cdot \gamma_s)_\beta.$$

Inequality (9) is thus satisfied for $M \geq M_1 \cdot B$. The second condition implies inequality (10). Indeed, for each edge γ_j labeled by α in $\mathcal{L}_2$, we have, using (K) in G,

$$(q \cdot \gamma_1 \dots \gamma_j)_\alpha - (q \cdot \gamma_1 \dots \gamma_{j-1})_\alpha \leq M \cdot \sum_{\substack{\beta \in l(E_{v'}) \\ \beta \neq \alpha}} (q \cdot \gamma_1 \dots \gamma_{j-1})_\beta$$

where v' is the ending vertex of $\gamma_1 \dots \gamma_{j-1}$. For each β in the sum, we have by the second condition an index i such that

$$(q \cdot \gamma_1 \dots \gamma_{j-1})_\beta \leq (q \cdot \gamma_1 \dots \gamma_{i-1})_\beta \leq B \cdot \sum_{\alpha \notin \mathcal{L}} (q \cdot \gamma_1 \dots \gamma_i)_\alpha - (q \cdot \gamma_1 \dots \gamma_{i-1})_\alpha \leq B \cdot \sum_{\alpha \notin \mathcal{L}} (q \cdot \gamma)_\alpha - q_\alpha.$$

where the second inequality is a consequence of (H4). □

3 Simplicial model for linear memory random walks

In this section we study a particular case of vector memory random walk for which the action is linear. We will see that some of these random walks have an associated deterministic dynamical system, defined on intersections of simplices, that we call *simplicial systems*. The properties of these two objects are strongly intertwined.

Definition 3.1. *Assume that for all edges e in the graph G, there exists a matrix $M_e \in \mathcal{M}_{|\mathcal{A}| \times |\mathcal{A}|}(\mathbb{R}_+)$ of determinant ± 1 such that for all $q \in \mathbb{R}_+^{\mathcal{A}}$ and all path $\gamma = e_1 \cdot e_2 \cdot \dots \cdot e_n$, the action on q can be expressed as*

$$q \cdot \gamma = q \cdot \underbrace{M_{e_1} \cdot M_{e_2} \cdot \dots \cdot M_{e_n}}_{M_\gamma}.$$

We call the induced random walk on G a linear memory random walk.

Recall the notation of projectivized quotient space PA associated to a linear space A from Section 1.1.

Definition 3.2. *Let $H = \{H_v\}_{v \in V}$ be a family of linear subspaces such that for all edge $e : v \to v'$,*

$$M_e H_{v'} = H_v.$$

We say that a linear memory random walk has a simplicial model *in the invariant family H, if for all vertex $v \in V$ of the graph there exists a set $\Delta_v \subseteq P(\mathbb{R}_+^{\mathcal{A}} \cap H)$ such that*

$$\forall\, e, e' \in E_v,\, e \neq e' \implies M_e \Delta_{e \cdot v} \cap M_{e'} \Delta_{e' \cdot v} = \emptyset \quad \text{and} \quad \bigsqcup_{e \in E_v} M_e \Delta_{e \cdot v} \subset \Delta_v.$$

Many models exist where H is the whole space. It is the case for all continued fraction algorithms described in [Fou25b]. We will sometimes denote by $\Delta_{\mathbf{w}}^H$ sets of a simplicial model to insist on its invariant linear subspaces family, and by d_H the dimension of these subspaces. A concrete example of a non trivial invariant linear subspaces family is given by Rauzy–Veech induction on linear involutions, for which the sum of the lengths on the top interval is always equal to the sum on the bottom interval.

When a linear memory random walk has a simplicial model, one considers the following parameter space

$$\Delta^\infty(G) = \bigsqcup_{v\in V} \{v\} \times \left(\bigcap_{n\in\mathbb{N}} \bigcup_{\substack{\gamma\in\Pi(v)\\ |\gamma|=n}} M_\gamma \Delta_{\gamma\cdot v} \right).$$

If $(v,\lambda) \in \Delta^\infty(G)$, there exists a unique edge $e : v \to v'$ such that $\lambda \in M_e\Delta_{v'}$. We thus define the map $T : \Delta^\infty(G) \to \Delta^\infty(G)$ by

$$T(v,\lambda) = (v', M_e^{-1}\lambda).$$

Then, to each point in $\Delta^\infty(G)$ corresponds a unique infinite path in the graph G. The subset of points starting with a given finite path $\gamma^* : v \to v'$ is

$$\Delta_{\gamma^*} := (\{v\} \times M_{\gamma^*}\Delta_{v'}) \cap \Delta^\infty(G).$$

We denote by $\Delta^*(G)$ the set of points in $\Delta^\infty(G)$ which visit Δ_{γ^*} infinitely many times or, in other words, points which associated path contains infinitely many copies of γ^* as factors. We can then define the first return map of T on the set $\Delta^* := \Delta^*(G) \cap \Delta_{\gamma^*}$,

$$T_* : \Delta^* \to \Delta^*.$$

An illustrative example shedding light on these simplicial models is the fractal set constructed as a subset of parameters within a simplex, where a win-lose induction remains confined to a subgraph. This fractal set possesses a simplicial model by construction, with a subsimplex associated to each finite path. As the length of paths increases, the union of subsimplices covers progressively less of the entire simplex. Ultimately, this process yields a fractal set with a smaller Hausdorff dimension in the limit.

In the following, we study the link between the probabilistic behavior of such linear memory random walk endowed with a simplicial model and the dynamical properties of T_*.

3.1 A uniformly expanding acceleration

For any two vectors $v, w \in \mathbb{R}_+^{\mathcal{A}}$, let

$$\alpha(v,w) := \max_{a\in\mathcal{A}} \frac{v_a}{w_a}, \qquad \beta(v,w) := \min_{a\in\mathcal{A}} \frac{v_a}{w_a}$$

and

$$d(v,w) := \log \frac{\alpha(v,w)}{\beta(v,w)}.$$

One can check that d is a complete metric on the projectivization of $\mathbb{R}_+^{\mathcal{A}}$ called the *Hilbert metric*. This metric has the useful feature that any linear map induced by a positive matrix is contracting with respect to it.

Proposition 3.3. *For any non-negative matrix M, we have for all $v, w \in \mathbb{R}_+^{\mathcal{A}}$,*

$$d(Mv, Mw) \leq d(v, w).$$

Moreover, if M is positive, there exists $\theta < 1$ such that

$$d(Mv, Mw) \leq \theta d(v, w).$$

Proof. This is a well known property of Hilbert metrics, the proof can be found *e.g.* in Section 2.1 of [Via97]. □

Hence the map T is expanding with respect to Hilbert metric but not strictly if the corresponding matrix is not positive. We designate a path γ^* in G as positive if M_{γ^*} is a positive matrix.

Proposition 3.4. *If γ^* is positive, there exists $\theta^* < 1$ such that for all $x, y \in \Delta^*$ which have the same path in G until they come back to $\Delta^*(G)$,*

$$d(T_* x, T_* y) \geq \frac{1}{\theta^*} d(x, y).$$

We say that T_* is uniformly expanding.

Proof of Proposition 3.4. Let $\gamma^* \cdot \gamma_t$ be the path in G associated to a given point in $\Delta^*(G)$ until its first return. The inverse branch of T along this path is a projectivization of the linear map $M_{\gamma^*} M_{\gamma_t}$, which is, according to Proposition 3.3, the composition of a weakly contracting map and a contracting map with coefficient $\theta^* < 1$ for the Hilbert metric on Δ. Hence the inverse branch is contracting by a coefficient θ^* depending only on γ^*. □

We now prove that the non-degenerating property implies the existence of a positive path.

Proposition 3.5. *If a vector memory random walk is non-degenerating, then for all subset of labels $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and from any vertex v there exists a finite path starting at v such that a letter in $\overline{\mathcal{L}}$ loses against a letter in $\mathcal{L}$ at some step.*

Proof. Consider $\mathscr{C}$ the strongly connected component of $G_{\mathcal{L}}$ to which v belongs. We start by following a path to a minimal strongly connected component $\mathscr{C}$. As in Remark 2.21, the non-degenerating property implies that $\mathscr{C}$ cannot be a minimal strongly connected component in G. Thus, there exists an edge in G going out of $\mathscr{C}$ on which a letter in $\mathcal{L}'$ loses against a letter in $\mathcal{L}$. □

Lemma 3.6. *A non-degenerating linear memory random walk admits a positive path starting from any vertex.*

Proof. By assumption (H1), coefficients of M_γ are non-decreasing. And according to (H4), for all vertex v with (at least) two outgoing edges labeled by α and β, we have

$$(M_\alpha)_{\beta,\alpha} \geq \frac{1}{B}.$$

In particular, to prove that $(M_\gamma)_{\alpha,\beta} > 0$, is it enough to show that β loses against α in γ.

Let $\alpha \in \mathcal{A}$ and v be a vertex of the graph. We show by induction that for all $n \leq |\mathcal{A}|$, there exist distinct letters $\beta_1, \beta_2, \ldots, \beta_n$ and a path γ_n starting at v such that for all $1 \leq i \leq n$, $(M_{\gamma_n})_{\alpha,\beta_i} > 0$. Assume it holds for some $1 \leq n < |\mathcal{A}|$. For $\mathcal{L} = \{\beta_1, \ldots, \beta_n\}$ there exists, by the previous proposition, γ' starting at $v \cdot \gamma_n$ such that a letter $\beta_{n+1} \notin \mathcal{L}$ loses against a letter in $\mathcal{L}$. The matrix $M_{\gamma_n \cdot \gamma'}$ is then positive at line α and columns $\beta_1, \ldots, \beta_{n+1}$.

We obtain a positive path by composing these paths constructed for each letter $\alpha \in \mathcal{A}$. □

3.2 Thermodynamic formalism

For the remainder of this text, we make several structural assumptions on the vector memory random walk that add up to Standing Assumptions 2.1 on its base graph.

Standing Assumptions 3.7.

- *The vector memory random walk is linear and has a simplicial model.*
- *The random walk is quickly escaping.*
- *There exists a positive path γ^* in the graph.*

By Corollary 2.20 and Lemma 3.6, the last two assumptions are automatically satisfied if the graph is non-degenerating.

Fix once and for all a positive path γ^* in the graph. We then delve into the dynamics of the associated first return map $T_* : \Delta^* \to \Delta^*$ which is uniformly expanding. The map T_* has countably many inverse branches which we label by

$$\mathcal{S} := \{w \text{ path from } \gamma^* \cdot v \text{ to } v \text{ in } G \mid w\gamma^* \text{ contains } \gamma^* \text{ only once as a factor}\}.$$

For $w \in \mathcal{S}$, let us define

$$\Delta_w := M_{\gamma^* w \gamma^*} \Delta_{\gamma^* \cdot v}.$$

It is the subsimplex of Δ^* for which the coding in T starts with path $\gamma^* w \gamma^*$ and goes back to Δ^*. The corresponding inverse branch sends $\Delta_w \cap \Delta^*$ to Δ^* by the matrix $M_{\gamma^* w}$. The sets $\Delta_w \cap \Delta^*$ form a partition of Δ^*.

Lemma 3.8. *The map T_* is conjugated on Δ^* to the full shift on $\Sigma = \mathcal{S}^{\mathbb{N}}$ by a homeomorphism.*

Proof. Let us consider $x \in \Delta^*$ and its corresponding infinite path $\overline{\boldsymbol{\gamma}}$ in G starting at v where γ^* appears infinitely many times. One can decompose $\overline{\boldsymbol{\gamma}} = \gamma^* w_1 \gamma^* w_2 \gamma^* w_3 \ldots$ and associate to $\overline{\boldsymbol{\gamma}}$ the infinite word on the alphabet $\mathcal{S}$, $w_1 w_2 w_3 \ldots$. By construction, the map T_* acts as the full shift on that infinite word.

Conversely, let us assume that we are given an infinite word in $\mathcal{S}$, $w_1 w_2 w_3 \ldots$. Let the sequence $(\Delta_n)_{n \geq 0}$ be defined by

$$\Delta_n = M_{\gamma^* w_1 \gamma^* \ldots \gamma^* w_n \gamma^*} \Delta_v.$$

By definition Δ_n is the set of points in Δ^* which coding for T starts with $\gamma^* w_1 \gamma^* w_2 \gamma^* \ldots w_n \gamma^*$. For every $n \geq 0$,

$$\Delta_{n+1} = M_{\gamma^* w_1 \gamma^* \ldots \gamma^* w_n \gamma^*} M_{w_{n+1} \gamma^*} \Delta_v$$

and as γ^* is a positive path $M_{w_{n+1}\gamma^*} \Delta_v = M_{w_{n+1}} M_{\gamma^*} \Delta_v$ is compactly included in Δ_v thus Δ_{n+1} is compactly included in Δ_n. Hence the set $\bigcap_{n=0}^{\infty} \Delta_n$ is non-empty, included in Δ^* and is reduced to a point again by positivity of M_{γ^*}. This point defines the inverse of the conjugacy map. Continuity of the map and its inverse are obvious using the Hilbert metric. □

To every cylinder of the shift $\mathbf{w} = [w_1, \ldots, w_n]$ we associate the set of points with the corresponding coding and the simplex $\Delta_{\mathbf{w}} := M_{\gamma^* w_1 \gamma^* \ldots \gamma^* w_n \gamma^*} \Delta_{\gamma^* \cdot v}$. The cylinder $\mathbf{w}$ is then in bijection with the set $\Delta_{\mathbf{w}} \cap \Delta^*$ with respect to the conjugacy.

On each vector space of the invariant family H, one can consider the Lebesgue measure. We denote its restriction on Δ^* by Leb, normalized such that $\mathrm{Leb}(\Delta_{\gamma^*}) = 1$.

Remark 3.9. *For win-lose induction, this measure corresponds to the measure on Δ^* induced by transporting the probability measure $\mathbb{P}_q^v$ with $q = (1, \ldots, 1)$ (denoted by $\mathbf{1}$) through the coding.*

Definition 3.10. *We denote by ν the measure $\mathbb{P}_{\mathbf{1}}^v$ pulled back by the coding to Δ^*.*

The goal of this section is to use thermodynamic formalism with a *geometric potential* measuring the Jacobian of T in order to understand ergodic properties of the measure ν for a simplicial system. To do so, we need the following hypothesis relating it with Lebesgue measure, similarly to the case of win-lose induction.

(Leb_*^H) There exists $C > 0$ such that for all $w \in \mathcal{S}$,

$$\frac{1}{C} \cdot \mathbb{P}_{\mathbf{1}}^v(\gamma^* w \gamma^*) \leq \mathrm{Leb}(\Delta_w) \leq C \cdot \mathbb{P}_{\mathbf{1}}^v(\gamma^* w \gamma^*).$$

In some examples, such as for win-lose induction, the bounds are satisfied more generally for every path $\gamma \in \Pi(v)$. Notice that this property depends on the extra structure given by the stable linear subspaces H since they are used to define the Lebesgue measure.

3.2.1 Roof function

Let us consider the *roof function* defined for all $x \in \Delta^\infty(G)$ by

$$r(x) = -\log \frac{|M_e^{-1} x|}{|x|}$$

where $e : v \to v'$ is the unique edge such that $x \in \{v\} \times M_e \Delta_{v'}$. And let the *accelerated roof function* be defined, for $x \in \Delta^*$, by

$$r_*(x) = r(x) + r(Tx) + \cdots + r(T^{n-1}x) = -\log \frac{|M_\gamma^{-1} x|}{|x|}$$

where $n \geq 1$ is the smallest integer such that $T^n x \in \Delta^*$ and γ is the finite path in the graph which is the coding of x until it returns to Δ^*. The path $\gamma = \gamma^* \gamma'$ is uniquely defined by the property that the coding of x starts with $\gamma^* \gamma' \gamma^*$ and γ^* appears only once as a factor of $\gamma' \gamma^*$.

In the above formulas we make the abuse of using the same notation for $x \in \Delta^\infty(G)$ and a representative of its class before projectivization. We can do so since the formulas are invariant by multiplication of x by a scalar in $\mathbb{R}_+$.

Let $0 < \theta^* < 1$ be the contracting constant associated to the matrix M_{γ^*} as in Proposition 3.4. We show that the accelerated roof function is *locally (or weakly) Hölder continuous with parameter θ^**.

Proposition 3.11. *For all $x, y \in \Delta^*$ in the same n-cylinder $\Delta_{\mathbf{w}}$, where $n \geq 1$,*

$$|r_*(x) - r_*(y)| \leq (\theta^*)^{n-1} \cdot \mathrm{diam}(\Delta^*).$$

Proof. Let $x, y \in \Delta^*$ be in the same cylinder $\Delta_{\mathbf{w}}$ where $\mathbf{w} = w_1 w_2 \ldots w_n$, then

$$|r_*(x) - r_*(y)| = \left| \log \frac{|M_{\gamma^* w_1}^{-1} y|}{|M_{\gamma^* w_1}^{-1} x|} \right|.$$

Notice that for any vector $v, w \in \mathbb{R}_+^{\mathcal{A}}$,

$$\frac{|v|}{|w|} \leq \frac{\alpha(v, w)}{\beta(v, w)}.$$

Thus by definition of Hilbert norm

$$|r_*(x) - r_*(y)| \le d(M_{\gamma^* w_1}^{-1} x,\, M_{\gamma^* w_1}^{-1} y) = d(T_* x,\, T_* y). \tag{11}$$

Let $x'' = T_*^n x$ and $y'' = T_*^n y$ in Δ^*, using Proposition 3.3,

$$\begin{aligned} d(T_* x,\, T_* y) &= d(M_{\gamma^*} M_{w_2} \dots M_{\gamma^*} M_{w_n} x'',\, M_{\gamma^*} M_{w_2} \dots M_{\gamma^*} M_{w_n} y'') \\ &\le \theta^* \cdot d(M_{w_2} M_{\gamma^*} M_{w_3} \dots M_{\gamma^*} M_{w_n} x'',\, M_{w_2} M_{\gamma^*} M_{w_3} \dots M_{\gamma^*} M_{w_n} y'') \\ &\le \theta^* \cdot d(T_*^2 x,\, T_*^2 y) \le (\theta^*)^{n-1} \cdot d(x'', y'') \le (\theta^*)^{n-1} \cdot \operatorname{diam}(\Delta^*). \end{aligned}$$

□

In the following lemma we prove a key property on the roof function to apply many theorems of thermodynamic formalism. The measure ν was introduced in Definition 3.10.

Lemma 3.12. *The accelerated roof function r_* has* exponential tail, *i.e. there exists $\sigma > 0$ such that*

$$\int_{\Delta^*} e^{\sigma r_*} \mathrm{d}\nu < \infty.$$

Proof. If $x \in \mathbb{R}_+^{\mathcal{A}}$, and γ is some path in the graph then

$$|M_\gamma x| = \sum_i \sum_j (M_\gamma)_{i,j} \cdot x_j \le \left(\max_j \sum_i (M_\gamma)_{i,j} \right) \cdot |x| = (\max \mathbf{1} \cdot M_\gamma) \cdot |x| \,.$$

Hence for x in the set $\{x \in \Delta^* \mid r_*(x) \ge \log \tau\}$ and γ the path corresponding to its first return to Δ^*, applying the above inequality to $M_\gamma^{-1} x$,

$$\tau \cdot \max \mathbf{1} \le \frac{|x|}{|M_\gamma^{-1} x|} \le \max \mathbf{1} \cdot M_\gamma. \tag{12}$$

The set is thus included in the subset of points in Δ^* that satisfy the property $\mathcal{J}^\tau \le \mathcal{E}_{\gamma^*}$ for initial distortion $\mathbf{1}$. By Corollary 2.10, there exists $C, \eta > 0$ such that, for all $\tau > 1$, $\nu\left(\{x \in \Delta^* \mid r_*(x) \ge \log \tau\}\right) \le C \tau^{-\eta}$. Then for a fixed $\tau > 1$, we split the integral into domains indexed by $n \in \mathbb{N}$, where $\log \tau^n \le r_*(x) < \log \tau^{n+1}$. For all $\sigma < \eta$,

$$\int_{\Delta^*} e^{\sigma r_*} \mathrm{d}\nu \le \sum_{n=0}^{\infty} (\tau^{n+1})^\sigma \cdot C \cdot (\tau^n)^{-\eta} = C \cdot \tau^\sigma \cdot \sum_{n=0}^{\infty} (\tau^{\sigma - \eta})^n = C \cdot \frac{\tau^\sigma}{1 - \tau^{\sigma - \eta}} < \infty.$$

□

Definition 3.13. *We denote by σ_0 the supremum of σ for which there is finiteness.*

As $e^{\sigma r_*}$ is positive and increasing in σ, the integral for $\sigma = \sigma_0$ is infinite by monotone convergence.

3.2.2 Estimates on the Jacobian

For A a $m \times m$ non-negative matrix of determinant ± 1, we denote the induced transformation on the positive cone by $\mathscr{L}_A : P\mathbb{R}_+^m \to P\mathbb{R}_+^m$. If H is a linear subspace of $\mathbb{R}^m$ such that the intersection of H and AH with $\mathbb{R}_+^{\mathcal{A}}$ have non-empty interior we denote by $\mathscr{L}_A^H : P(H \cap \mathbb{R}_+^{\mathcal{A}}) \to P(AH \cap \mathbb{R}_+^{\mathcal{A}})$ the restriction of $\mathscr{L}_A$ to these spaces.

The map T_* is locally of this form. We list here some useful properties on the Jacobian determinant of such maps, denoted $\mathbb{J}_A$ and $\mathbb{J}_A^H$ respectively, and relate it to the roof function.

By an elementary computation that can be found *e.g.* in [Vee78], Proposition 5.2, one has the following formulas.

Proposition 3.14. *For all $x \in P\mathbb{R}_+^m$,*

$$\mathbb{J}_A(x) = \left(\frac{|x|}{|Ax|}\right)^m$$

and for all linear subspace $H \subset \mathbb{R}^m$,

$$\mathbb{J}_A^H(x) = \left(\frac{|x|}{|Ax|}\right)^{d_H}$$

where d_H is short for $\dim H$.

Assume f is equal in the neighborhood of x to a projective linear map $\mathscr{L}_A$. We use the following notations $Df(x) := \mathbb{J}_A(x)$ and $D^H f(x) := \mathbb{J}_A^H(x)$.
In particular, for all $x \in \Delta^*$,

$$DT_*(x) = e^{|\mathcal{A}| \cdot r_*(x)} \quad \text{and} \quad D^H T_*(x) = e^{d_H \cdot r_*(x)}.$$

Remark 3.15. *In the following, for the sake of readability, we prove results in the case $H = \mathbb{R}^{|\mathcal{A}|}$ but all the arguments are the same for other H where we replace $|\mathcal{A}|$ by d_H, DT_* by $D^H T_*$ and $\Delta_{\mathbf{w}}$ by $\Delta_{\mathbf{w}}^H$.*

Proposition 3.16. *There exists a decreasing sequence $Q_n = 1 + o(1) > 0$, such that for all $n \in \mathbb{N}$, all finite word $\mathbf{w} = w_1 \cdot \dots \cdot w_n$ in $\mathcal{S}$ and all $x \in \Delta_{\mathbf{w}}$*

$$\frac{1}{Q_n} \cdot DT_*^n(x)^{-1} \leq \mathrm{Leb}(\Delta_{\mathbf{w}}) \leq Q_n \cdot DT_*^n(x)^{-1}.$$

In particular, there exists a constant $Q > 0$ such that for any two finite words $\mathbf{w}_1$ and $\mathbf{w}_2$,

$$\frac{1}{Q} \leq \frac{\mathrm{Leb}(\Delta_{\mathbf{w}_1 \mathbf{w}_2})}{\mathrm{Leb}(\Delta_{\mathbf{w}_1}) \,\mathrm{Leb}(\Delta_{\mathbf{w}_2})} \leq Q.$$

Remark 3.17. *This second inequality corresponds to the bounded distortion property of [AF07] in the case of Rauzy–Veech induction.*

Proof. Let x, y be both in $\Delta_{\mathbf{w}}$ and let $M_{\mathbf{w}} = M_{\gamma^* w_1 \gamma^* \dots \gamma^* w_n}$, then $M_{\mathbf{w}}^{-1} x, M_{\mathbf{w}}^{-1} y \in M_{\gamma^*}\Delta$ and, as in (11), $d(M_{\mathbf{w}}^{-1} x, M_{\mathbf{w}}^{-1} y) \leq (\theta^*)^{n-1} \cdot \mathrm{diam}(\Delta^*)$. This implies the existence of $Q_n = 1 + o(1) > 0$, depending only on $\mathrm{diam}(M_{\gamma^*}\Delta)$ and θ^*, such that

$$\frac{1}{Q_n} \cdot \mathbb{J}_{M_{\mathbf{w}}^{-1}}(x) \leq \mathbb{J}_{M_{\mathbf{w}}^{-1}}(y) \leq Q_n \cdot \mathbb{J}_{M_{\mathbf{w}}^{-1}}(x). \tag{13}$$

By integrating these inequalities for y in $\Delta_{\mathbf{w}}$,

$$\frac{1}{Q_n} \cdot \mathbb{J}_{M_{\mathbf{w}}^{-1}}(x) \cdot \mathrm{Leb}(\Delta_{\mathbf{w}}) \leq \mathrm{Leb}(M_{\gamma^*}\Delta) \leq Q_n \cdot \mathbb{J}_{M_{\mathbf{w}}^{-1}}(x) \cdot \mathrm{Leb}(\Delta_{\mathbf{w}}).$$

Then for all $\mathbf{w}$ and all $x \in \Delta_{\mathbf{w}}$

$$\frac{1}{Q_n} \cdot \mathrm{Leb}(\Delta_{\mathbf{w}})^{-1} \leq \mathbb{J}_{M_{\mathbf{w}}^{-1}}(x) \leq Q_n \cdot \mathrm{Leb}(\Delta_{\mathbf{w}})^{-1}. \tag{14}$$

The map T_*^n coincides with $\mathscr{L}_{M_{\mathbf{w}}^{-1}} : \Delta_{\mathbf{w}} \to M_{\gamma^*}\Delta$ on this domain, implying the first inequality of the proposition.

Consider now inequalities (14) for the word $\mathbf{w}_1$ and integrate them for x in $\Delta_{\mathbf{w}_1\mathbf{w}_2}$. We obtain in the middle part a change of variable formula for $y = M_{\mathbf{w}_1}^{-1}x$ which varies in the domain $\Delta_{\mathbf{w}_2}$.

$$\frac{1}{Q_0} \cdot \mathrm{Leb}(\Delta_{\mathbf{w}_1})^{-1} \cdot \mathrm{Leb}(\Delta_{\mathbf{w}_1\mathbf{w}_2}) \leq \mathrm{Leb}(\Delta_{\mathbf{w}_2}) \leq Q_0 \cdot \mathrm{Leb}(\Delta_{\mathbf{w}_1})^{-1} \cdot \mathrm{Leb}(\Delta_{\mathbf{w}_1\mathbf{w}_2}).$$

□

Corollary 3.18. *There exists $Q > 0$ such that for all $w \in \mathcal{S}$, corresponding to a path of length n, every $\kappa > 0$ and any $x \in \Delta_w$*

$$\frac{1}{Q} \cdot \mathrm{Leb}(\Delta_w)^{\kappa/|\mathcal{A}|} \leq e^{-\kappa\left(r_*(x)+\cdots+r_*(T_*^{n-1}x)\right)} \leq Q \cdot \mathrm{Leb}(\Delta_w)^{\kappa/|\mathcal{A}|}.$$

With invariant subspaces

$$\frac{1}{Q} \cdot \mathrm{Leb}\left(\Delta_w\right)^{\kappa/d_H} \leq e^{-\kappa\left(r_*(x)+\cdots+r_*(T_*^{n-1}x)\right)} \leq Q \cdot \mathrm{Leb}\left(\Delta_w\right)^{\kappa/d_H}.$$

We write $A \simeq B$ when there exists a constant K such that $1/K \cdot B \leq A \leq K \cdot A$ independently of the variables of A and B. Hypothesis (Leb_*^H) can be stated as, for all $w \in \mathcal{S}$, $\mathrm{Leb}\left(\Delta_w\right) \simeq \mathbb{P}_{\mathbf{1}}^v\left(\gamma^* w \gamma^*\right)$. Using the previous corollary, we have for all $x \in \Delta_w$,

$$e^{|\mathcal{A}|\cdot r_*(x)} \simeq \frac{1}{\mathbb{P}_{\mathbf{1}}^v\left(\gamma^* w \gamma^*\right)}.$$

Hence, the integral of this function on Δ^* must be infinite. Which implies,

Proposition 3.19. *Under hypothesis (Leb_*^H), $\sigma_0 \leq |\mathcal{A}|$.*

3.2.3 Invariant measure equivalent to Lebesgue

Using such control on the Jacobian we prove the existence of a unique invariant measure equivalent to Lebesgue for a quickly escaping simplicial system.

Proposition 3.20. *If (Leb_*^H) is satisfied, there exists a unique ergodic T_*-invariant Borel probability measure μ absolutely continuous with respect to Lebesgue measure. Moreover, the logarithm of its density $|\log \frac{d\mu}{d\,\mathrm{Leb}}|$ is bounded by a global constant at almost every point.*

As T_* is a first return map of T, if we have control on the Jacobian of T outside of Δ^* (as is the case for win-lose inductions), an ergodic measure for T_* which is absolutely continuous with respect to Lebesgue measure on Δ^* induces an ergodic measure with the same regularity for T on Δ. This measure can be either finite or infinite on Δ, as is the case for Brun and Gauss maps respectively [Fou25b].

Proof. For all path $\gamma = \gamma^* \cdot w_1 \cdot \gamma^* \cdot \cdots \cdot \gamma^* \cdot w_n \cdot \gamma^*$ with $w_1, \ldots, w_n \in \mathcal{S}$,

$$\mathbb{P}_{\mathbf{1}}^v\left(\gamma \cdot \bullet \mid \gamma\right) = \mathbb{P}_q^{v'}$$

where $q = \mathbf{1} \cdot \gamma \in \mathbb{R}_+^{\mathcal{A}}$ and $v' = \gamma \cdot v$. But the path γ^* acts on distortion vectors by a positive matrix which ratio of coefficients is bounded by a constant D depending only on γ^*; thus q is D-balanced. We have

$$\frac{\mathrm{Leb}(M_\gamma A)}{\mathrm{Leb}(\Delta_\gamma^H)} \simeq \mathrm{Leb}(A).$$

Moreover,

$$T_*^{-n}A = \bigcup_{w_1,\dots,w_n\in\mathcal{S}} M_\gamma A.$$

Thus

$$\mathrm{Leb}(T_*^{-n}A) \simeq \sum_{w_1,\dots,w_n\in\mathcal{S}} \mathrm{Leb}(A)\cdot\mathrm{Leb}(\Delta_\gamma^H) = \mathrm{Leb}(A).$$

Hence the sequence of probability measures $\frac{1}{n}\sum_{i=0}^{n-1}(T_*^i)_*\,\mathrm{Leb}$ is bounded and there exists an extraction that converges to an invariant probability measure μ with bounded log-density.

Ergodicity is a classical consequence of bounded distortion property of Proposition 3.16 and unicity follows directly . Let $[\mathbf{w}_1] = [w_1,\dots,w_n]$ be a cylinder for the shift on $\mathcal{S}^{\mathbb{N}}$, we define the positive measure on cylinders

$$[\mathbf{w}_2] \mapsto \mathrm{Leb}(\Delta_{\mathbf{w}_1}\cap T_*^{-n}\Delta_{\mathbf{w}_2}) - \frac{1}{Q}\cdot\mathrm{Leb}(\Delta_{\mathbf{w}_1})\cdot\mathrm{Leb}(\Delta_{\mathbf{w}_2}).$$

As the cylinders generate the Borel σ-algebra, for any T_*-invariant Borel set A, the measure defined on cylinders by

$$[\mathbf{w}_1] \mapsto \mathrm{Leb}(\Delta_{\mathbf{w}_1}\cap A) - \frac{1}{Q}\cdot\mathrm{Leb}(\Delta_{\mathbf{w}_1})\cdot\mathrm{Leb}(A)$$

is positive. And so is

$$\mathrm{Leb}\Big((\mathcal{S}^{\mathbb{N}}\setminus A)\cap A\Big) - \frac{1}{Q}\cdot\mathrm{Leb}(\mathcal{S}^{\mathbb{N}}\setminus A)\cdot\mathrm{Leb}(A) \geq 0$$

which implies that $\mathrm{Leb}(A)=0$ or $\mathrm{Leb}(\mathcal{S}^{\mathbb{N}}\setminus A)=0$. □

3.2.4 Gibbs measures and Gurevic–Sarig pressure

We continue by investigating a wide class of invariant probability measures coming from thermodynamic formalism. The following definitions are usually introduced in a more general context where T_* is a shift on the space of Markov chain Σ, instead of a brave full shift on $\Sigma = \mathcal{S}^{\mathbb{N}}$ here.

Definition. *Let μ be a T_*-invariant Borel probability measure, for any continuous function $\phi : \Sigma \to \mathbb{R}$, μ will be called a* Gibbs measure for the potential ϕ *if there exist $Q>0$ and P in $\mathbb{R}$ such that for every x in the cylinder $[w_1,\dots,w_n]$*

$$\frac{1}{Q} \leq \frac{\mu([w_1,\dots,w_n])}{\exp\left(\sum_{k=0}^{n-1}\phi(T_*^k(x)) - Pn\right)} \leq Q. \tag{15}$$

When ϕ is Hölder-continuous there exists a unique such measure and a unique such P which is called the topological pressure of ϕ (see Theorem 3.1 in [Pes14]).

In the following, we consider the potential functions $\phi_\kappa = -\kappa r_*$ for $\kappa \geq 0$. When there is no ambiguity we will denote one of these functions simply by ϕ. We demonstrate that they possess favorable properties for inducing the existence and uniqueness of Gibbs measures.

The Ruelle operator $L_\phi : C(\Delta^*) \to C(\Delta^*)$ associated to a potential function ϕ is an operator acting on the space of continuous functions from Δ^* to $\mathbb{R}$. For a function $f \in C(\Delta^*)$ it is defined by

$$(L_\phi f)(x) = \sum_{T_*(y)=x} e^{\phi(y)} f(y).$$

As explained in [Sar15]: "the analysis of thermodynamic limits reduces to the study of the asymptotic behavior of $L_\phi^n f$ as $n \to \infty$ for *sufficiently many* functions f ". One of the key to understand this behavior is to first understand the limit of $\frac{1}{n} \log L_\phi^n f$. In particular, it can be compared to the following quantities.

For $w \in \mathcal{S}$, let us define the sum on periodic points

$$Z_n(\phi, w) = \sum_{T_*^n(x)=x,\, x_0=w} e^{\phi_n(x)}$$

with $\phi_n = \phi + \phi \circ T_* + \cdots + \phi \circ T_*^{n-1}$. According to Theorem 4.3 in [Sar15], when ϕ has summable variations, the limit

$$P(\phi) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, w) \tag{16}$$

exists for all $w \in \mathcal{S}$ and is independent of w. Moreover, if $\|L_\phi 1\|_\infty < \infty$, then $P(\phi) < \infty$.

Definition. *$P(\phi)$ is called the Gurevic–Sarig pressure of ϕ.*

This is a relevant quantity to consider according to Theorem 4.4 of [Sar15] since, when $P(\phi)$ is finite, it is equal to the limit of $\frac{1}{n} \log L_\phi^n f$ for a large class of functions. It is not always the case for characteristic functions $\chi_{[w]}$ for which we only have an upper bound.

Remark 3.21. *As a consequence of Sarig's Generalized Ruelle–Perron–Frobenius Theorem, in [Sar15], for all $w \in \mathcal{S}$ and $x \in \Delta^*$*

$$P(\phi) \geq \limsup_{n\to\infty} \frac{1}{n} \log \left(L_\phi^n \chi_{[w]}\right)(x).$$

Definition. *The potential function ϕ has summable variations if and only if*

$$\sum_{n=2}^{\infty} var_n(\phi) < \infty,$$

where $var_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x_i = y_i, i = 1, \ldots, n\}$.

Notice that the Hölder property proved in Proposition 3.11 implies that, for all κ, ϕ has summable variations and $var_1(\phi) < \infty$.

These definitions enable us to state the key theorem of this section. It gives a criterion for uniqueness of a Gibbs measure for a given potential function. The following formulation of Sarig theorem is taken from Theorem 4.6 in [Pes14]. The BIP property for Markov chains it mentions is clearly satisfied in our case since we are considering a full shift. Existence is due to Sarig [Sar03] and uniqueness to Buzzi–Sarig [BS03]. Finiteness of entropy is a consequence of Theorem 5.5 in [Sar15].

Theorem. *Assume that the potential ϕ has summable variations. The potential ϕ admits a unique Gibbs measure μ_ϕ if and only if the Gurevic–Sarig pressure and $var_1(\phi)$ are finite.*

In this case, topological and Gurevic–Sarig pressures coincide, ϕ is positive recurrent and the Gibbs measure is the Ruelle–Perron–Frobenius measure of ϕ. This measure is the unique equilibrium measure and it has finite entropy.

In our setting, it implies this simpler statement.

Corollary 3.22. *When $P(\phi) < \infty$ there exists a unique Gibbs measure μ_ϕ of potential ϕ. It is the unique equilibrium measure for ϕ and has finite entropy.*

We study finiteness of the pressure by relating it to exponential tail integrals. Let I_σ stand for $\int_{\Delta^*} e^{\sigma r_*} \mathrm{d}\nu$. We have seen that $I_\sigma < \infty$ if and only if $\sigma < \sigma_0$.

Proposition 3.23. *Under assumption (*Leb_*^H*), there exists a constant K such that for all $\sigma < \sigma_0$*

$$\frac{1}{K} \cdot I_\sigma \leq \sum_{w \in \mathcal{S}} e^{-(|\mathcal{A}|-\sigma)\cdot r_*(w)} \leq K \cdot I_\sigma$$

and for $\sigma \geq \sigma_0$,

$$\sum_{w \in \mathcal{S}} e^{-(|\mathcal{A}|-\sigma)\cdot r_*(w)} = +\infty.$$

In the formulas, $r_*(w)$ stands for $r_*(x)$ at some point in $x \in w$. The proposition does not depend on these choices according to Proposition 3.11.

Proof. As before, we write $A \simeq B$ when there exists a constant K such that $1/K \cdot B \leq A \leq K \cdot A$ for all σ and choices of $x \in w$. Corollary 3.18, assumption (Leb_*^H) and Proposition 3.11 show respectively that, for all $w \in \mathcal{S}$,

$$e^{-(|\mathcal{A}|-\sigma)\cdot r_*(w)} \simeq \mathrm{Leb}(\Delta_w) \cdot e^{\sigma \cdot r_*(w)} \simeq \mathbb{P}_{\mathbf{1}}^v(\gamma^* w \gamma^*) \cdot e^{\sigma \cdot r_*(w)} \simeq \int_{\Delta_w} e^{\sigma r_*} \mathrm{d}\nu$$

We end the proof by summing over $\mathcal{S}$. □

This link enables us to characterize the values of κ for which the pressure is finite.

Lemma 3.24. *If a linear random walk satisfies (*Leb_*^H*) then its pressure $P(\phi_\kappa)$ is finite if and only if $\kappa > |\mathcal{A}| - \sigma_0$.*

Proof. By Formula (13), $L_\phi 1 = \sum_{T_*(y)=x} e^{\phi_\kappa(y)} \leq (Q')^{\kappa/|\mathcal{A}|} \cdot \sum_{w \in \mathcal{S}} e^{-\kappa r_*(w)}$. Which is finite for $|\mathcal{A}| - \kappa < \sigma_0$. And as mentioned above, finiteness of $P(\phi_\kappa)$ is implied by finiteness of $L_\phi 1$.

Assume now that the pressure if finite. Then according to Remark 3.21, we have $\limsup_{n\to\infty} \frac{1}{n} \log\left(L_\phi^n \chi_{[w]}\right)(x) < +\infty$. But then

$$\begin{aligned} L_\phi^n \chi_{[w]}(x) &= \sum_{T_*^n(y)=x} e^{-\kappa \cdot r_*{}^{(n)}(y)} \chi_{[w]}(y) \\ &\geq (Q')^{-n\cdot\kappa/|\mathcal{A}|} \sum_{w_1,\ldots,w_{n-1} \in \mathcal{S}} e^{-\kappa \cdot r_*(w)} e^{-\kappa \cdot r_*(w_1)} \ldots e^{-\kappa \cdot r_*(w_{n-1})} \\ &= (Q')^{-n\cdot\kappa/|\mathcal{A}|} \cdot e^{-\kappa \cdot r_*(w)} \cdot \left(\sum_{w_1 \in \mathcal{S}} e^{-\kappa \cdot r_*(w_1)}\right)^n \end{aligned}$$

Hence $\sum_{w_1 \in \mathcal{S}} e^{\kappa \cdot r_*(w_1)}$ must be finite and $|\mathcal{A}| - \kappa < \sigma_0$. □

The proposition has also the following important consequence.

Proposition 3.25. *Under assumption (*Leb_*^H*),* $P(\phi_\kappa) \xrightarrow[\kappa\to+\infty]{} -\infty$.

Proof. Let us take $\kappa = |\mathcal{A}| - \left(\sigma - \frac{\kappa_1}{m}\right)$, where $\sigma < \sigma_0$ and m is a lower bound for r_*, then

$$Z_n(\phi, w) = \sum_{T_*^n(x)=x,\, x_0=w} e^{\phi_n(x)} \leq Q' \cdot \sum_{w_1,\dots,w_n\in\mathcal{S}} e^{-\kappa(r_*(w_1)+\cdots+r_*(w_n))}$$
$$\leq Q' \cdot \left(K \cdot I_{\sigma-\frac{\kappa_1}{m}}\right)^n \leq Q' \cdot (K\cdot I_\sigma)^n \cdot e^{-n\kappa_1}.$$

Thus $\lim \frac{1}{n}\log Z_n(\phi, n) \leq \log(K\cdot I_\sigma) - \kappa_1$ and letting κ_1 go to infinity we obtain the result. □

On the other hand, by (12), the roof function is bounded away from zero, hence,

Proposition 3.26. $P(\phi_\kappa) \xrightarrow[\kappa\to-\infty]{} +\infty$.

3.3 Suspension flow

Let us define the *suspension space* by $\Delta_r^* := (\Delta^*\times\mathbb{R})/\sim$ where for all $(x,t)\in\Delta^*\times\mathbb{R}$ we have $(x,t)\sim\left(\widehat{T}_*x, t+r_*(x)\right)$. The associated *suspension semi-flow* is defined on Δ_r^*, for all $t\geq 0$, by

$$\Phi_t^* : (x,s)\to(x,s+t).$$

Notice that this semi-flow is defined such that the first return map to the section $\Delta^*\times\{0\}$ is T_* and its return time is r_*.

To have a flow, one need to define an invertible extension of our map. Let us denote by $\widehat{\Delta}^*$ the set of bi-infinite words in $\mathcal{S}$. The shift on this space is denoted by $\widehat{T}_*$ and is semi-conjugated to T_*. It is invertible and is called the *natural extension* of T_*.

On can now define similarly the *suspension space* $\widehat{\Delta}_r^*$. And its associated *suspension flow* on $\widehat{\Delta}_r^*$ denoted again by Φ_t^*.

Metrics Let us define for $x,y\in\mathcal{S}^{\mathbb{N}}$ the metric

$$\delta(x,y) = (\theta^*)^{\ell(x,y)}$$

where $\ell = \min\{k\geq 0 \mid x_k\neq y_k\}$ and θ^* is taken from Proposition 3.4, such that the Hilbert metric of the corresponding points of the simplex in Δ^* with the given coding for T_* satisfy $d(x,y)\leq \mathrm{diam}(\Delta^*)\cdot\delta(x,y)$. Where the diameter is taken for distance d. In particular, Hölder functions for d are also Hölder for δ.

This metric can be extended to $\widehat{\Delta}^* := \mathcal{S}^{\mathbb{Z}}$ by considering

$$\ell = \min\{k\geq 0 \mid x_{-k}\dots x_k \neq y_{-k}\dots y_k\}.$$

On $\widehat{\Delta}_r^*$, we consider the product metric induced by δ and the euclidean metric in fibers.

Measures Denote by $\mathcal{M}_{T_*,r_*}$ the set of T_*-invariant Borel probability measures with $\mu(r_*) := \int_{\Delta^*} r_* d\mu < +\infty$. Notice that to each T_*-invariant measure μ, one can associate a unique $\widehat{T}_*$-invariant measure which extends it.

Every Φ^*-invariant Borel probability measure $\widetilde{\mu}$ on $\widehat{\Delta}^*_r$ can be decomposed as a product of a measure $\mu \in \mathcal{M}_{T_*,r_*}$ (extended to $\widehat{\Delta}^*$) and the Lebesgue measure on fibers. Namely,

$$\widetilde{\mu}_{r_*} = (\mu(r_*))^{-1} (\mu \times \mathrm{Leb})_{|\widehat{\Delta}^*_r}.$$

The Kolmogorov–Sinai entropy of the flow for this measure is written $h(\Phi^*, \widetilde{\mu})$ and satisfies Abramov's formula

$$h(\Phi^*, \widetilde{\mu}) = \frac{h(T_*, \mu)}{\mu(r_*)}$$

where $h(T_*, \mu)$ is the Kolmogorov–Sinai entropy for T_*. In this setting, the topological entropy can be defined as

$$h_{\mathrm{top}}(\Phi^*) = \sup_{\mu \in \mathcal{M}_{T_*,r_*}} h(\Phi^*, \widetilde{\mu}_{r_*}).$$

An induced measure $\widetilde{\mu}_{r_*}$ for $\mu \in \mathcal{M}_{T_*,r_*}$ at which this supremum is achieved (and by extension μ itself) is referred to as a *measure of maximal entropy.*

By our Standing Assumptions 2.1 and 3.7, the coding map in Proposition 1.1 is injective on $\Delta^\infty(G)$ and conjugates the map T up to zero measure subsets to a subshift on a subset of paths in G . We define the natural extension of T as the subshift on bi-infinite paths in G and a suspension flow Φ with the map r.

Proposition 3.27. *The flows Φ^* and Φ are conjugated and their exponential tail integrals are equal.*

This is a fundamental remark, as it demonstrates that the flow Φ^* does not depend on the choice of γ^*. While we initially focus on studying Φ^* due to the favorable dynamical properties of T_*, the ultimate goal is to analyze Φ. This suspension is thus referred to as the *canonical suspension* of $\Delta^\infty(G)$.

Proposition 3.28. *Under assumption (*Leb^H_**), the pressure $P(\phi_\kappa)$ vanishes at a unique value $\kappa_0 > |\mathcal{A}| - \sigma_0$. The measure induced by μ_ϕ is the unique measure of maximal entropy for the canonical suspension flow, with entropy equal to κ_0.*

Notice, in particular, that this provides an intrinsic definition of κ_0 as the topological entropy of the suspension flow does not depend on the choice of γ^*.

Proof. The map $\kappa \mapsto P(\phi_\kappa)$ is a convex, decreasing function of κ (see Theorem 4.6 of [Sar15]). Utilizing Lemma 3.24, Proposition 3.25, and Proposition 3.26, we establish the first fact by continuity of this function.

According to Corollary 3.22, there exists a Gibbs measure μ of finite entropy for this vanishing value of κ. This measure is an equilibrium measure and satisfies the *variational principle* for the topological pressure (see Section 5.3 in [Sar15]). Consequently, μ maximizes the quantity

$$h(T_*, \mu) - \int_{\Delta^\infty(G)} \kappa_0 r_* \mathrm{d}\mu, \tag{17}$$

over all Borel measures such that $\mu(\phi) > -\infty$ (i.e., $\mu(r_*) < +\infty$). Its maximum value is equal to the pressure, here 0. This also implies that $h(\Phi^*, \widetilde{\mu}) = \frac{h(T_*,\mu)}{\mu(r_*)} = \kappa_0$ is maximal. According to Theorem 1.1 in [BS03], there is at most one such maximizing measure. □

3.3.1 Exponential mixing and Central limit theorem

Consider an observable $\phi : \Delta^* \to \mathbb{R}$, let us show that it has exponential decay of correlation with any other observable. Classically, this can be reformulated as showing exponential decay of the following norm.

$$\|E(\phi|\mathcal{F}_n)\|_2 := \sup\left\{\int \xi\phi \mathrm{d}\mu \mid \xi \in L^2(\mathcal{F}_n) \text{ and } \|\xi\|_2 = 1\right\}.$$

Where $\mathcal{F}_n = {r_*}^{-n}(\mathcal{B})$ is the preimage of the Borel σ-algebra.

Define ϕ to be weakly r_*, α-Hölder on Δ^* if there is a constant C such that for all $x, x' \in \mathcal{S}^{\mathbb{N}}$ with $x_0 = x'_0$ we have

$$|\phi(x) - \phi(x')| \leq C \cdot \sup_{\Delta_{x_0}} r_* \cdot \delta(x, x')^\alpha.$$

Let us denote by $C_\alpha(\phi)$ the smallest such constant for a r_*, α-Hölder fonction. We denote this set of functions by $H^\alpha(\Delta^*)$. As we have seen, the metric δ can be extended to $\widehat{\Delta}^*$ and we denote by $H^\alpha(\widehat{\Delta}^*)$ the set of weakly r_*, α-Hölder function on this set.

Proposition 3.29. *If $\phi \in H^\alpha(\Delta^*) \cap L^p(\Delta^*)$ of $\phi \in H^\alpha(\widehat{\Delta}^*) \cap L^p(\widehat{\Delta}^*)$ for some $p > 1$ then there exists $C, \sigma > 0$ such that for all $n \geq 0$*

$$\|E(\phi|\mathcal{F}_n) - E(\phi)\|_2 \leq C \cdot (C_\alpha(\phi) + \|\phi\|_p) \cdot e^{-\sigma n}.$$

Proof. Let $\Delta_k = \{x \in \Delta^* \mid r_*(x) \leq k\}$ and consider the decomposition $\phi = \phi_1 + \phi_2 + \phi_3$ defined by

$$\phi_1 = \sum_{\mathbf{w} \in \mathcal{S}^n} \phi(x_\mathbf{w}) \chi_{\Delta_\mathbf{w}} \chi_{\Delta_k}, \quad \phi_2 = \phi\chi_{\Delta_k} - \phi_1 \quad \text{and} \quad \phi_3 = \phi \cdot (1 - \chi_{\Delta_k}).$$

Where $x_\mathbf{w}$ is an arbitrary element of $\Delta_\mathbf{w}$. Notice that ϕ_1 is independent of $\mathcal{F}_n$, thus

$$E(\phi_1|\mathcal{F}_n) = E(\phi_1).$$

By Proposition 3.11, there exists a constant C_1 such that for all $x \in \Delta_\mathbf{w}$,

$$\begin{aligned} |\phi_2(x)| &= |\phi(x) - \phi(x_\mathbf{w})| \\ &\leq C_\alpha(\phi) \cdot k \cdot \delta(x, x_\mathbf{w})^\alpha \\ &\leq C_\alpha(\phi) \cdot C_1 \cdot k \cdot (\theta^*)^{\alpha n} \end{aligned}$$

By Hölder inequality, we have for all $p \in (1, \infty]$,

$$\|\phi_3\|_2 = E(|\phi \cdot (1 - \chi_{\Delta_k})|^2) \leq \|\phi\|_p \cdot E(|1 - \chi_{\Delta_k}|)$$

and $\mathbb{P}(r_*(x) \geq k) \leq I_\sigma \cdot e^{-\sigma k}$. Thus there exists C_2 such that

$$\|\phi_3\|_2 \leq C_2 \cdot \|\phi\|_p \cdot e^{-\sigma k}.$$

The same argument also holds for $\|\phi_3\|_1$. Hence, applying these remarks, there exists $\sigma > 0$ such that for all k and n we have

$$\begin{aligned} \|E(\phi|\mathcal{F}_n) - E(\phi)\|_2 &\leq |E(\phi) - E(\phi_1)| + \|E(\phi_2|\mathcal{F}_n)\|_2 + \|E(\phi_3|\mathcal{F}_n)\|_2 \\ &\leq \|\phi_2\|_1 + \|\phi_3\|_1 + \|\phi_2\|_2 + \|\phi_3\|_2 \\ &\leq 2C_1 \cdot C_\alpha(\phi) \cdot k \cdot e^{-\sigma n} + 2C_2 \cdot e^{-\sigma k} \end{aligned}$$

We conclude by taking $k = n$ and σ a bit smaller to get rid of the linear factor. □

This implies exponential decay of correlation.

Corollary 3.30. *Let $p > 1$ then, for any $\alpha > 0$, there exists positive constants C, δ such that for any $\phi \in H^\alpha(\Delta^*) \cap L^p(\Delta^*, \mu)$ and $\psi \in L^2(\Delta^*, \mu)$ we have*

$$\left| \int \phi \cdot \psi \circ T_*^n \, \mathrm{d}\mu - \int \phi \, \mathrm{d}\mu \int \psi \, \mathrm{d}\mu \right| \leq C \cdot \big(C_\alpha(\phi) + \|\phi\|_p \big) \cdot \|\psi\|_2.$$

If $p > 2$ and both $\phi, \psi \in H^\alpha(\widehat{\Delta}^*) \cap L^p(\widehat{\Delta}^*, \mu)$ then

$$\left| \int \phi \cdot \psi \circ \widehat{T}_*^n \, \mathrm{d}\mu - \int \phi \, \mathrm{d}\mu \int \psi \, \mathrm{d}\mu \right| \leq C \cdot \big(C_\alpha(\phi) + \|\phi\|_p \big) \cdot \big(C_\alpha(\psi) + \|\psi\|_p \big).$$

Another consequence is that for such ϕ, if $\int \phi \, \mathrm{d}\mu = 0$, then $\sum_{n=1}^\infty \|E(\phi|\mathcal{F}_n)\|_2 < \infty$. Which classically (see Theorem 2.11 in [Via97]) implies a Central Limit Theorem (CLT).

Corollary 3.31. *Let $\phi \in H^\alpha(\Delta^*) \cap L^p(\Delta^*, \mu)$ with $\int \phi \, \mathrm{d}\mu = 0$. Assume that there does not exist $\psi \in L^2(\Delta^*, \mu)$ such that $\phi = \psi \circ T_* - \psi$. Then there exists $\sigma_\phi > 0$ such that*

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \phi \circ T_*^n \xrightarrow{d} \mathcal{N}(0, \sigma_\phi) \quad \text{as } N \to \infty.$$

Where the convergence is in distribution to a normal law of variance σ_ϕ.

The same result holds on $\widehat{\Delta}^*$ with $p > 2$.

One can then go from central limit theorem on T_* to the canonical suspension, by applying the main theorem in [MT04]. To apply this theorem, we need to check the criterion in the CLT for the roof function. First, notice that by exponential tail property, $r_* \in L^p(\Delta^*)$ for any $p > 0$. Moreover, we show in the following lemma that it is not a coboundary.

Lemma 3.32. *If the graph contains two positive loops, such that the invariant direction of their associated matrix are not projectively equal, then it is not possible to write $r_* = \psi + \phi \circ T_* - \phi$ where ψ is constant on each tile Δ_w of Δ^* with $w \in \mathcal{S}$ and ϕ in L^2.*

Proof. Assume we have such an expression of r_*. Consider an element of Δ^* represented by vectors such that $|x| = 1$. Differentiating (in the sense of distributions) ${r_*}^{(n)}(x) = r_* \circ T^{n-1}(x) + \cdots + r_*(x)$ we have $D({r_*}^{(n)}) = D(\phi \circ T_*^n) - D(\phi)$ and for h^n an inverse branch of T_*^n, $D({r_*}^{(n)} \circ h^n) = D(\phi) - D(\phi \circ h^n)$. Now notice that we can rewrite $r_*(x) = -\log |M^{-1}x|$ and ${r_*}^{(n)} \circ h^n(x) = \log |M_n x|$ where M is the non-negative matrix associated to the path of the tile in which x is taken and M_n the matrix associated to the inverse branch. Thus for $v \in \mathbb{R}^{\mathcal{A}}$,

$$Dr_*(x) \cdot v = \frac{|M_n v|}{|M_n x|}.$$

Notice that $D(\phi \circ h^n) \to 0$ thus for all $v, w \in \Delta^*$,

$$\frac{|M_n v|}{|M_n w|} = \frac{\langle (1, \ldots, 1) M_n, \, v \rangle}{\langle (1, \ldots, 1) M_n, \, w \rangle}$$

admits a limits independent of the choice of inverse branch. In particular, this implies that $(1, \ldots, 1) M_n$ converges projectively to a limit independent of the path. This contradicts the hypothesis on the two loops. ∎

This implies that $r_* - \overline{r_*}$ satisfies the above CLT and thus the following theorem. Let us denote by $H^\alpha(\widehat{\Delta}_r^*)$ the space of Hölder functions for the induced product metric on $\widehat{\Delta}_r^*$.

We have the following dictionary between weakly r, α-Hölder functions on the spaces $\widehat{\Delta}^*$ and α-Hölder functions on its suspension space $\widehat{\Delta}_r^*$. Its proof is straight forward and left to the reader.

Proposition 3.33. *Consider $\varphi \in H_\alpha(\widehat{\Delta}_r^*)$ and $\phi : \Delta^* \to \mathbb{R}$ defined for $x \in \Delta^*$ by*

$$\phi(x) = \int_0^{r_*(x)} \varphi\left(\Phi_t(x)\right) \mathrm{d}t.$$

Then ϕ is a weak r_*, α-Hölder function on Δ^* and for $p > 2$, if $\varphi \in L^p(\widehat{\Delta}_r^*, \mu_r)$ then $\phi \in L^p(\widehat{\Delta}^*, \mu)$. Moreover, if there exists $\psi \in L^2(\widehat{\Delta}^*, \mu)$ such that $\phi = \psi \circ T_* - \psi$ then there exists $\widetilde{\psi} \in L^2(\widehat{\Delta}_r^*, \mu_r)$ differentiable in the direction of the suspension such that $\varphi = X_t \widetilde{\psi}$, where X_t is the derivative in that direction.

This enables us to state a CLT on the canonical suspension flow. For convenience in the notation, we consider the suspension of $\widehat{\Delta}^*$ but recall that this suspension is intrinsic to the simplicial model by Proposition 3.27.

Theorem 3.34 (Central Limit Theorem)**.** *Let $p > 2$ and let $\varphi \in H^\alpha(\widehat{\Delta}_r^*) \cap L^p(\widehat{\Delta}_r^*, \mu_r)$ satisfy $\int \varphi \, \mathrm{d}\mu_r = 0$. Assume that there does not exist $\widetilde{\psi} \in L^2(\widehat{\Delta}_r^*, \mu_r)$ differentiable in the direction of the suspension such that $\varphi = X_t \widetilde{\psi}$ where X_t is the Lie derivative in that direction. Then there is a positive constant σ_ϕ such that*

$$\frac{1}{\sqrt{|T|}} \int_0^T \varphi \circ \Phi_t \, dt \xrightarrow{d} \mathcal{N}(0, \sigma_\phi) \quad \text{as } |T| \to \infty.$$

3.4 Bound on Hausdorff dimension

We consider the Gibbs equilibrium measure μ — *i.e.* maximizing quantity (17) — associated to κ_0. According to Formula (15) in the definition of Gibbs measures, there exists $Q > 0$ such that, for all x in the cylinder $\mathbf{w} = [w_1, \ldots, w_n]$,

$$\frac{1}{Q} \cdot \exp\left(- \sum_{k=0}^{m-1} \kappa_0 \cdot r_*(T_*^k(x)) \right) \leq \mu(\mathbf{w}) \leq Q \cdot \exp\left(- \sum_{k=0}^{m-1} \kappa_0 \cdot r_*(T_*^k(x)) \right).$$

Remark 3.35. *As we did in a previous subsection, we describe the case of $H = \mathbb{R}^{|\mathcal{A}|}$ but all the arguments are the same for other H where we replace $|\mathcal{A}|$ by d_H, DT_* by $D^H T_*$ and $\Delta_{\mathbf{w}}$ by $\Delta_{\mathbf{w}}^H$.*

Corollary 3.18 implies that,

$$\exp\left(-|\mathcal{A}| \sum_{k=0}^{m-1} r_*(T_*^k(x)) \right) \simeq \mathrm{Leb}(\Delta_{\mathbf{w}}).$$

Thus, for all cylinders $\mathbf{w}$,

$$\mu(\mathbf{w}) \simeq \mathrm{Leb}(\Delta_{\mathbf{w}})^{\kappa_0/|\mathcal{A}|}. \tag{18}$$

Proposition 3.36. *There exists $K > 0$ such that for all simplex Δ of dimension d, measure m and diameter less than 1, the minimal number of balls of radius $0 < \rho \leq m$ required to cover Δ satisfies*

$$N_\rho \leq K \cdot \frac{m}{\rho^d}.$$

For $m = \rho$ this implies in particular $N_\rho \leq K \cdot \rho^{1-d}$.

Since T_* is uniformly expanding, for all $\epsilon > 0$, there exists $\mathcal{F}$ a family of cylinders that form a partition of the space of Δ^* and such that $\mathrm{Leb}(\Delta_{\mathbf{w}}) < \epsilon$. Then, for all $\mathbf{w} \in \mathcal{F}$ one can find a covering $\{B_i^{\mathbf{w}}\}$ by less than $K \cdot \mathrm{Leb}\,(\Delta_{\mathbf{w}})^{1-d}$ balls of radius $\mathrm{Leb}\,(\Delta_{\mathbf{w}}) < \epsilon$. For this covering, we then have

$$\sum_{\mathbf{w}\in\mathcal{F}} \sum_i (\mathrm{diam}\, B_i^{\mathbf{w}})^{\delta} \leq \sum_{\mathbf{w}\in\mathcal{F}} K \cdot \mathrm{Leb}\,(\Delta_{\mathbf{w}})^{1-d} \cdot \mathrm{Leb}\,(\Delta_{\mathbf{w}})^{\delta}.$$

By Formula (18),

$$\sum_{\mathbf{w}\in\mathcal{F}} \mathrm{Leb}\,(\Delta_{\mathbf{w}})^{\kappa_0/|\mathcal{A}|} \simeq \sum_{\mathbf{w}\in\mathcal{F}} \mu\,(\mathbf{w}) = \mu\left(\bigcup_{\mathbf{w}\in\mathcal{F}} \mathbf{w}\right) = 1.$$

Thus if $1 - d + \delta \geq \kappa_0/|\mathcal{A}|$ then $\sum_{\mathbf{w}} \sum_i (\mathrm{diam}\, B_i^{\mathbf{w}})^{\delta}$ is bounded uniformly for all ϵ. Then

$$\dim_H \Delta^* \leq d - 1 + \kappa_0/|\mathcal{A}|. \tag{19}$$

To induce an inequality on the whole space $\Delta^\infty(G)$, we need to control the Haussdorff dimension of the set of parameters which coding starts with γ^* but for which this path never appears again.

Proposition 3.37. *Let $\Delta^\infty := \Delta_{\gamma^*} \cap \Delta^\infty(G)$. Under assumption ($\mathrm{Leb}_*^H$),*

$$\dim_H \Delta^\infty \setminus \Delta^* \leq |\mathcal{A}| - 1 - \frac{\sigma_0}{|\mathcal{A}|}.$$

Proof. By assumption (H2), the sum of r along an orbit by T of a point in Δ^∞ goes to infinity. Thus, for any $x_0 \in \Delta^\infty$ and all $\epsilon > 0$, there exists a finite path starting at $\gamma^* \cdot v$ such that $x_0 \in \Delta_{\gamma^*\cdot\gamma}$, $\mathrm{Leb}\,(\Delta_{\gamma^*\cdot\gamma}) \leq \epsilon$ and for all $x \in \Delta_{\gamma^*\cdot\gamma}$ we have $\mathbb{J}_{M^{-1}_{\gamma^*\cdot\gamma}}(x) > 1/\epsilon$.

This implies the existence of a countable covering of the simplex by such sets. Moreover, if two simplices $\Delta_{\gamma^*\cdot\gamma}$ and $\Delta_{\gamma^*\cdot\gamma'}$ overlap, it means one of the path is prefix from the other. Thus, this covering can be assumed to be a partition. Denote by Γ the corresponding set of paths γ labeling the partition.

For all $\gamma \in \Gamma$, there exists a covering $\{B_i^\gamma\}$ of $\Delta_{\gamma^*\cdot\gamma}$ by less than $K{\cdot}\mathrm{Leb}\,(\Delta_{\gamma^*\cdot\gamma})^{1-d}$ balls of radius $\mathrm{Leb}\,(\Delta_{\gamma^*\cdot\gamma})$. And

$$\sum_{\gamma\in\Gamma}\sum_i (\mathrm{diam}\, B_i^\gamma)^\delta \leq \sum_{\gamma\in\Gamma} K \cdot \mathrm{Leb}\,(\Delta_{\gamma^*\cdot\gamma})^{1-d} \cdot \mathrm{Leb}\,(\Delta_{\gamma^*\cdot\gamma})^{\delta} \leq C \cdot K \cdot \epsilon^{\delta-d} \cdot \sum_{\gamma\in\Gamma} \mathbb{P}_1^v(\gamma^*\cdot\gamma).$$

The condition on the Jacobian for simplices of the cover implies that for all $x \in \Delta_{\gamma^*\cdot\gamma}\cap\Delta^*$, $e^{|\mathcal{A}|\cdot r_*(x)} > 1/\epsilon$. Hence $\sum_{\gamma\in\Gamma} \mathbb{P}_1^v(\gamma^*\cdot\gamma) \leq \nu\left(e^{|\mathcal{A}|\cdot r_*(x)} > 1/\epsilon\right)$. Moreover for $\sigma < \sigma_0$ the tail integral I_σ is finite and $\nu\left(e^{|\mathcal{A}|\cdot r_*(x)} > 1/\epsilon\right) < I_\sigma \cdot \epsilon^{-\frac{\sigma}{|\mathcal{A}|}}$. Thus, we have the bound

$$\sum_{\gamma\in\Gamma}\sum_i (\mathrm{diam}\, B_i^\gamma)^\delta \leq C \cdot K \cdot I_\sigma \cdot \epsilon^{d-\delta-\frac{\sigma}{|\mathcal{A}|}}.$$

which is bounded by a constant independent of ϵ for $\delta \leq d - \frac{\sigma}{|\mathcal{A}|}$. Hence $\dim_H \Delta^\infty \setminus \Delta^* \leq d - \frac{\sigma}{|\mathcal{A}|}$ for all $\sigma < \sigma_0$ and thus $\dim_H \Delta^\infty \setminus \Delta^* \leq d - \frac{\sigma_0}{|\mathcal{A}|}$. ☐

Recall that T is locally projective linear and its Jacobian is bounded away from 0 and ∞; thus the image sets $T^n\Delta^*$ have the same Hausdorff dimension. In particular, $\Delta^*(G)$ which is a countable union of such sets has the same Hausdorff dimension. And the same occurs for $\Delta^\infty(G)$ and $\Delta^\infty(G) \setminus \Delta^*(G)$. Equation (19) and Proposition 3.37 thus imply the following.

Theorem 3.38. *Under assumption* (Leb$_*^H$)*, the Hausdorff dimension of* $\Delta^\infty(G)$ *is bounded by*

$$\dim_H \Delta^\infty(G) \leq |\mathcal{A}| - 1 - \frac{|\mathcal{A}| - \kappa_0}{|\mathcal{A}|}.$$

If the simplices are included in a family of invariant subspace H*,*

$$\dim_H \Delta^\infty(G) \leq d_H - 1 - \frac{d_H - \kappa_0}{d_H}.$$

Hausdorff dimension for subgraphs Consider a linear memory random walk on a graph G with a simplicial model and F a subgraph of G. In this paragraph, standing assumptions will only be made on the linear memory random walk on F induced by G.

Standing Assumptions 3.39.

- *F is finite and strongly connected.*
- *The action of paths on distortion vectors satisfies hypothesis (H1-4) and* (Leb$_*^H$)*.*
- *F is non-degenerating or admits a non-degenerating factorization as defined respectively in Definition 2.22 and Definition 2.28.*

By Lemma 3.6, there is a positive path γ^* in F which enables us to define the first return map T_*.

The simplicial model on G induces a simplicial model on F. We denote by $\Delta^\infty(F,G)$ and $\Delta^*(F,G)$ its associated parameter sets, defined in the introduction of the section, to keep track of the fact that the simplicial model is induced by the one on G.

Corollary 3.40. *If there exists a vertex v in F which has an outgoing edge in G and not in F, then*

$$\dim_H \Delta^\infty(F,G) < d_H - 1.$$

Proof. We start by proving that the Lebesgue measure of $\Delta^*(F,G)$ is zero. As in the proof of Proposition 3.20, after γ^* the distortion is balanced, thus the probability that a path leaves F is bounded from below by some $\epsilon > 0$ depending only on γ^*. And the probability that γ^* appears n times in a path before leaving F is bounded from above by $(1-\epsilon)^n$. By (Leb$_*^H$) we then have

$$\sum_{\mathbf{w}\in\mathcal{S}^n} \mathrm{Leb}(\Delta_{\mathbf{w}}) \leq (1-\epsilon)^n.$$

Which proves that

$$\mathrm{Leb}(\Delta^*(F,G)) \leq \sum_{\mathbf{w}\in\mathcal{S}^n} \mathrm{Leb}(\Delta_{\mathbf{w}}) \xrightarrow[n\to\infty]{} 0.$$

Now, let $P = P(\phi_{d_H})$ and μ be the Gibbs measure for this potential. Then for all n-cylinder $\mathbf{w}$, we have

$$\mu(\mathbf{w}) \simeq \mathrm{Leb}(\Delta_{\mathbf{w}}) \cdot e^{-Pn}$$

up to a constant independent of n. Since, for all $n \geq 1$, $\sum_{\mathbf{w}\in\mathcal{S}^n} \mu(\mathbf{w}) = 1$, P must by negative. As the pressure is decreasing in κ, the vanishing value for κ satisfies $\kappa_0 < d_H$. Theorem 3.38 then implies $\dim_H \Delta^\infty(F,G) < d_H - 1$. □

3.5 Consequences on win-lose inductions

Win-lose inductions satisfy (Leb^H_*) due to the following stronger property:

(Leb^H) For all vertices v and all paths γ starting at v,

$$\frac{\mathrm{Leb}(\Delta^H_\gamma)}{\mathrm{Leb}(\Delta^H)} = \mathbb{P}^v_{\mathbf{1}}(\gamma).$$

This assumption generalizes the case of win-lose induction considered in Section 1.1 and is false in general for fractal sets constructed as parameters remaining in a subgraph, such as the Rauzy gasket.

Proposition 3.41. *If (*Leb^H*) is satified, the zero pressure parameter is* $\kappa_0 = d_H$.

Proof. We show that the measure μ from Proposition 3.20 is the unique Gibbs measure for potential $-d_H \cdot r_*$.

According to Corollary 3.18, there exists a constant $Q > 0$ such that, for all x in the cylinder $[x_1, \dots, x_n]$,

$$\frac{1}{Q} \leq \frac{\mathrm{Leb}([x_1, \dots, x_n])}{\exp\left(\sum_{k=0}^{n-1} -d_H \cdot r_*\left(T_*^k(x)\right)\right)} \leq Q.$$

As μ is such that $|\log \frac{d\mu}{d\,\mathrm{Leb}}|$ is bounded at almost every point then it satisfies the same property for another constant Q. Then μ is a Gibbs measure for the potential $-d_H \cdot r$ and it has zero topological pressure. By Proposition 3.28, the function $P(-\kappa \cdot r_*)$ vanishes at a unique value κ_0 which is then equal to d_H. □

We summarize these results in the following theorem. Recall this is under Standing Assumptions 2.1 and 3.7.

Theorem 3.42. *Under assumption (*Leb^H*), the canonical suspension flow has a unique measure of maximal entropy which is the suspension of the unique T-invariant Borel measure absolutely continuous with respect to Lebesgue measure. Moreover, the entropy for this measure is* d_H.

In particular, the only thing one needs to check on win-lose induction for the conclusion of this theorem to be true, is the graph criterion defined as the non-degenerating property.

Remark 3.43. *Notice that the measure suspension of the unique T-invariant measure absolutely continuous with respect to Lebesgue measure is finite. It is a consequence of the exponential tail property. For Rauzy–Veech induction, this measure corresponds to Masur–Veech measure (see [Fou25a] for details). This was one of the first key results of the theory of translation surfaces proved by Masur and Veech independently in 1982. The computation of these volumes has been an active research subject ever since and has seen major advances recently.*

Measure of maximal entropy as a limit According to Proposition 3.16, there exists a sequence $Q_n = 1 + o(1) > 0$ such that

$$\frac{1}{Q_n} \cdot \mathrm{Leb}([x_1, \dots, x_n]) \leq \exp\left(\sum_{k=0}^{n-1} -d_H \cdot r_*\left(T_*^k(x)\right)\right) \leq Q_n \cdot \mathrm{Leb}([x_1, \dots, x_n]).$$

Where x is chosen as the unique point of n-periodic coding starting with $x_1, \dots, x_n$. Summing on all cylinders, we get

$$\frac{1}{Q_n} \leq \sum_{T_*^n(x)=x} \exp\left(\sum_{k=0}^{n-1} -d_H \cdot r_*\left(T_*^k(x)\right)\right) \leq Q_n.$$

Then when $n \to 1$, it goes to 1 and there is no mass going to infinity for the Gibbs measure. Theorem 7.8 in [GS98] then implies the following.

Corollary 3.44. *If δ_x denotes the Dirac measure supported on x and μ the ergodic measure equivalent to Lebesgue on $\Delta^\infty(G)$,*

$$\mu = \lim_{n\to\infty} \frac{1}{P_n} \sum_{x\in \mathrm{Fix}(T_*^n)} e^{-d_H \cdot {r_*}^{(n)}(x)}\, \delta_x$$

where ${r_*}^{(n)} = r_* + r_* \circ T_* + \cdots + r_* \circ T_*^{n-1}$ and $P_n = \sum_{x\in \mathrm{Fix}(T_*^n)} e^{-d_H \cdot {r_*}^{(n)}(x)}$.

Strongly positive recurrence By the same argument, we have $\frac{1}{Q_n}\,\mathrm{Leb}([w]) \leq Z_n(\phi, w) \leq Q_n\,\mathrm{Leb}([w])$. And thus, $\sum_{n\geq 0} Z_n(\phi, w)$ diverges. By Theorem 6.10 in [Sar15], this implies that the discriminant is infinite thus by Theorem 6.7 in the same article, the following holds.

Proposition 3.45. *The potential $\phi = -d_H \cdot r_*$ is positive recurrent with spectral gap property.*

The Spectal Gap Property and its implications are defined in section 6 of that article. They imply in particular an exponential mixing and Central Limit Theorem for the shift (which are weaker in this setting than the one proved in Section 3.3.1.

Theorem 6.5 in [Sar15] establishes that the perturbed pressure is analytic in a neighbourhood of zero is the key ingredient for equidistribution theorem in [PP90]. All their construction using dynamical zeta function should work the same using this result. It will be done in details in an upcoming article. Up to checking this construction, this would imply the following conjectural result (as well as other finer counting estimates).

Conjecture. *For γ a periodic orbit of the suspension flow on $\widehat{\Delta}_r^*$, we denote by δ_γ the Dirac measure supported on it with total mass $\ell(\gamma)$ its length. Let μ_r be the unique measure of maximal entropy for the flow, then*

$$\frac{h}{e^{hL}} \cdot \sum_{\ell(\gamma)\leq L} \delta_\gamma \to \mu_r$$

where the convergence is for weak topology as L goes to infinity. Moreover, if $N(L)$ denotes the number of closed orbit of the flow of length less than L,

$$N(L) \sim \frac{e^{hL}}{hL}.$$

Notice that the last part of the result was proved in the case of strata of quadratic differentials [EMR19].

Acknowledgments

I am deeply grateful to Valérie Berthé, Vincent Delecroix, Sébastien Gouëzel, Pascal Hubert, Malo Jezequel, and Sasha Skripchenko for useful discussions related to this project. I also extend my thanks to Paul Mercat and Thierry Coulbois for their meticulous review of early versions of this article.

This material is based upon work supported by the National Science Foundation under Grant No. 1440140, while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the semester of fall 2019 *Holomorphic differentials in Mathematics and Physics*.

This work was supported by the Agence Nationale de la Recherche through the project Codys (ANR- 18-CE40-0007).

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