Ergodic properties of Rauzy–Veech inductions

Charles Fougeron

Abstract

We introduce a unified description of Rauzy–Veech induction on interval exchanges and linear involutions with or without flips using simplicial systems. This enables us to give a new straightforward and common proof of the existence and uniqueness of the measure of maximal entropy for Teichmüller flow on abelian and quadratic differentials strata. In both cases it corresponds to the classical Masur–Veech measure.

For interval exchanges and linear involutions with flips we obtain the existence of a periodic subinterval for almost every parameters as well as an upper bound on the Hausdorff dimension of the complementary set of such parameters. This strengthens the results of Nogueira Danthony–Nogueira and Skripchenko–Troubetzkoy.

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1 Definitions

1.1 Win-lose induction

Let $G = (V, E)$ denote a graph labeled on an alphabet $\mathcal{A}$ by a function $l : E \to \mathcal{A}$ such that all vertex $v \in V$ has either zero or two outgoing edges with distinct labels.. Moreover, for every $v \in V$, the restriction of l to E_v, the set of edges going out of v, is assumed to be injective.

Let V^0 be the set of vertices in V with no outgoing edges. A vertex v in $V \setminus V^0$ has by assumption two outgoing edges e, f respectively labeled by $\alpha, \beta \in \mathcal{A}$. The subcones

$$\mathcal{K}^e := \left\{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha < \lambda_\beta\right\} \quad \text{and} \quad \mathcal{K}^f := \left\{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\beta < \lambda_\alpha\right\}$$

form a partition of $\mathbb{R}_+^{\mathcal{A}}$ where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$. They have the same boundary set thus depending only on the vertex v

$$\mathcal{H}^v := \left\{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha = \lambda_\beta\right\}.$$

Additionally, we associate matrices

$$M_e := \text{Id } + E_{\beta,\alpha} \quad \text{and} \quad M_f := \text{Id } + E_{\alpha,\beta}.$$

Where $E_{a,b}$ is the elementary matrix with coefficient 1 at row a and column b. Such that $\mathcal{K}^e = M_e \cdot \mathbb{R}_+^{\mathcal{A}}$ and $\mathcal{K}^f = M_f \cdot \mathbb{R}_+^{\mathcal{A}}$. Hence it is natural to define

$$\Theta_e : \begin{cases} \mathcal{K}^e & \to \quad \mathbb{R}_+^{\mathcal{A}} \\ \lambda & \mapsto \quad M_e^{-1}\lambda \end{cases}.$$

The *win-lose induction* associated to the graph G is the map

$$\Theta : \left(V \setminus V^0\right) \times \mathbb{R}_+^{\mathcal{A}} \to V \times \mathbb{R}_+^{\mathcal{A}}$$

defined for every edge e from vertices v to v' and all $\lambda \in \mathcal{K}^e$ by $\Theta(v, \lambda) = (v', \Theta_e(\lambda))$.

Remark 1.1. *The map is only defined on*

$$\bigsqcup_{v \in V \setminus V^0} \left(\{v\} \times \mathbb{R}_+^{\mathcal{A}} \setminus \mathcal{H}^v\right)$$

but we make this abuse of notation for clarity, since these hyperplanes will not play a role in the Lebesgue generic dynamical behaviour nor the Hausdorff dimensions we will estimate.

Consider a vertex v with two or more outgoing edges and a parameter $\lambda \in \mathbb{R}_+^{\mathcal{A}}$. In analogy with Rauzy–Veech induction (for an introduction, refer to [Yoc10]), we call the edge e such that $\lambda \in \mathcal{K}^e$ the **loser**. Conversely, the labels of any other edge e' in E_v is called a **winner**, and we say it wins against e. We sometimes say a label wins or loses when there is no ambiguity to which edge they correspond.

The map Θ can be characterized as follows: it compares the coordinates of all edges emanating from a given vertex v on the vector and subtracts the smallest coordinate from the others, effectively subtracting the losing coordinate from the winning ones.

Remark 1.2. *In the following, we denote an edge by its label when there is no ambiguity. Using for instance $\mathcal{K}^\alpha$ instead of $\mathcal{K}^e$.*

Let us consider the projectivization relation $x \sim \lambda x$ satisfied for all $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}_+^{\mathcal{A}}$. We denote by $\Delta := \mathbb{R}_+^{\mathcal{A}} / \sim$ the simplex of dimension $|\mathcal{A}| - 1$ and, for each $v \in V$, by $\{\Delta_e\}_{e \in E_v}$ its induced partition by $\{\mathcal{K}^e\}_{e \in E_v}$. The maps Θ_e can be quotiented by this relation and we denote the induced map by $T_e : \Delta_e \to \Delta$. Similarly, Θ induces a map on space $\Delta(G) := V \times \Delta$ denoted by $T : \Delta(G) \to \Delta(G)$.

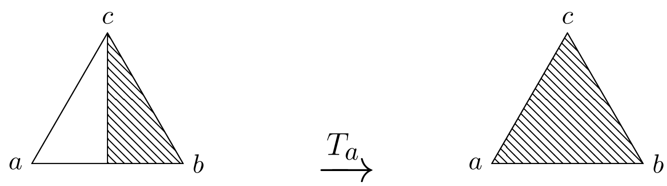


Figure 1: Action of T_a on Δ_a

Remark 1.3. *Classically (see [Yoc10]), Rauzy–Veech induction is represented with its Rauzy diagram, a graph with vertices labeled with the corresponding interval exchange permutation which edges are labeled by* top *of* bottom *depending on which interval* wins *in the induction and points to the corresponding new permutation. As a win-lose induction, we prefer to label the edges by the* losing label*. In Figure 2, we represent Rauzy diagram for a 3-IET with the labeling of its corresponding win-lose induction.*

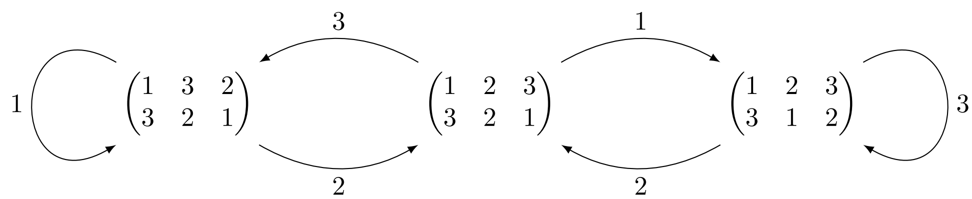


Figure 2: Rauzy diagram for 3-IET.

Let $F = \left(V^F, E^F\right)$ be a subgraph of a win-lose induction base graph G.

Definition 1.4. *A family of linear subspaces $H = \{H_v\}_{v\in V}$ is invariant for the subgraph F if for all edge $e : v \to v'$ in E^F*

$$M_e H_{v'} = H_v.$$

When we have such a invariant family, the win-lose induction satisfies for all edge $e : v \to v'$ of F, $\Theta_e(\mathcal{K}^e \cap H_v) = \mathbb{R}_+^{\mathcal{A}} \cap H_{v'}$. One can then restrict Θ to the parameter space

$$H(F, G) = \bigcup_{v\in V^F} \{v\} \times \left(\mathbb{R}_+^{\mathcal{A}} \cap H_v\right).$$

and define similarly

$$\Delta_H(F, G) = H(F, G)/ \sim .$$

Remark 1.5. *Theses concepts are useful to describe Rauzy–Veech induction for linear involutions. We describe a similar win-lose induction as for IETs, but for linear involutions there is a condition on top and bottom length to match that defines an invariant family of linear subsets. This family is invariant for a subgraph where some edges are forbidden, namely those that point to a permutation for which this condition cannot be true.*

The subgraph F is given by vertices on the top line and thick edges. This example and its generalizations will be explained in details in Section 2.

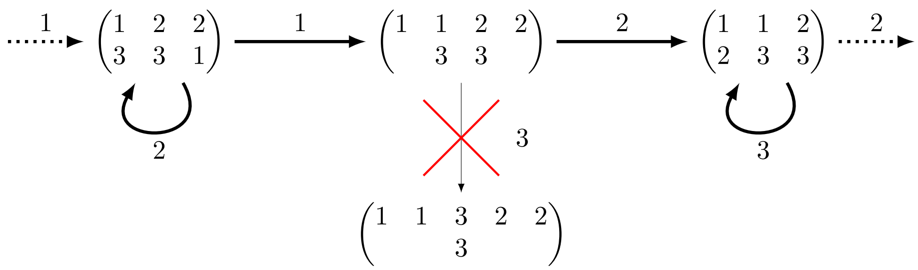


Figure 3: Piece of Rauzy diagram for linear involution on 3 intervals.

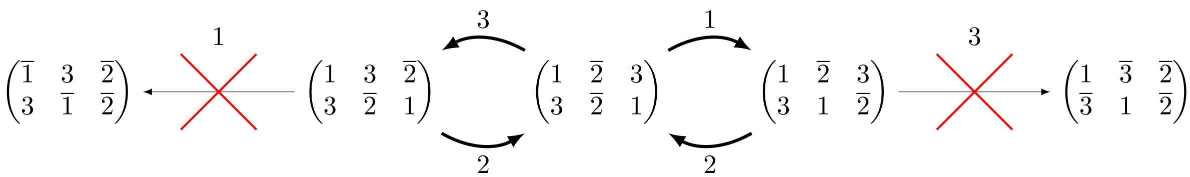


Figure 4: Rauzy diagram for 3-IET with a flip.

Roof function Let us define the *roof function* for (almost) all $x \in \Delta(G)$ as follows. Let e be the edge from vertex v to v' such that $x \in \{v\} \times M_e \Delta_{v'}$, we set

$$r(x) = -\log\left(\frac{\left|M_e^{-1}x\right|}{|x|}\right).$$

Suspension semi-flow Define the *suspension space* $\Delta(G)_r := (\Delta(G) \times \mathbb{R})/\sim$, where for all $(x,t) \in \Delta(G) \times \mathbb{R}$ we have the equivalence $(x,t) \sim (Tx, t + r(x))$. The associated *suspension semi-flow* is defined on $\Delta(G)_r$, for all $t \geq 0$, by

$$\phi_t : (x,s) \to (x, s+t).$$

Notice that this flow is defined such that the first return map to the section $\Delta(G) \times \{0\}$ is T and its return time is r.

Denote by $\mathcal{M}_{T,r}$ the set of T-invariant Borel probability measures with $\mu(r) := \int_{\Delta(G)} r d\mu < +\infty$. Every ϕ-invariant Borel probability measure $\widetilde{\mu}$ on $\Delta(G)_r$ can be decomposed as a product of a measure $\mu \in \mathcal{M}_{T,r}$ and the Lebesgue measure on fibers. Namely,

$$\widetilde{\mu}_r = (\mu(r))^{-1} (\mu \times \mathrm{Leb})_{|\Delta(G)_r}.$$

The Kolmogorov–Sinai entropy of the flow for this measure is written $h(\phi, \widetilde{\mu})$ and satisfies Abramov's formula

$$h(\phi, \widetilde{\mu}) = \frac{h(T,\mu)}{\mu(r)}$$

where $h(T,\mu)$ is the Kolmogorov–Sinai entropy for T. In this setting the topological entropy can be defined as

$$h_{\mathrm{top}}(\phi) = \sup_{\mu \in \mathcal{M}_{T,r}} h(\phi, \widetilde{\mu}_r).$$

The induced measure $\widetilde{\mu}_r$ for $\mu \in \mathcal{M}_{T,r}$ at which this supremum is achieved (and by extension μ itself) is referred to as a *measure of maximal entropy*.

Again the same suspension can be defined for a subgraph on $\Delta_H(F,G)_r := (\Delta_H(F,G) \times \mathbb{R})/\sim$.

1.2 Non-degenerating properties

1.2.1 Classical Rauzy diagrams

Assume that, on a non-trivial subset of labels $\mathcal{L} \subset \mathcal{A}$, the parameter in $\mathbb{R}_+^{\mathcal{A}}$ have coordinates in $\mathcal{L}$ infinitely smaller than others. At a vertex that with at least one outgoing edge labeled in $\mathcal{L}$, any edge labeled outside of $\mathcal{L}$ must win. Hence, the map T will remain in a subgraph in which we remove such edges not labeled in $\mathcal{L}$.

This motivates the introduction of the *degenerate subgraph* $G^{\mathcal{L}}$ having the same set of vertices V as G but for which we remove edges along which a letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$. For a vertex $v \in V$ in $G^{\mathcal{L}}$, the set of outgoing edges is defined as follows.

- If $l(E_v) \cap \mathcal{L} \neq \emptyset$
$$E_v^{\mathcal{L}} = \{e \in E_v \mid l(e) \in \mathcal{L}\}.$$
- Otherwise
$$E_v^{\mathcal{L}} = E_v.$$

Definition 1.6 (Non-degenerating graph)**.** *We say that the base graph of a vector memory random walk is* non-degenerating *if it is strongly connected and, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and all vertices v in a strongly connected component $\mathscr{C}$ of $G^{\mathcal{L}}$, one of the following properties holds:*

1. *There is a path from v in G labeled in $\mathcal{L}$ leaving $\mathscr{C}$.*
2. $|l(E_v) \cap \mathcal{L}| \leq 1$.

In plain words: from any vertex, no letter in $\mathcal{L}$ can win against another letter in $\mathcal{L}$ in any strongly connected component of $G^{\mathcal{L}}$ except if there is a path labeled in $\mathcal{L}$ leaving the component.

It is easy to check that this property is satisfied by a Rauzy diagrams associated to an irreducible IET (see Proposition 2.15 in [Fou24]). This implies many ergodic properties for Rauzy–Veech induction in this case, as a consequence of the following theorem in [Fou24].

Theorem A. *Every non-degenerating win-lose induction has a unique invariant measure equivalent to Lebesgue measure and it induces the unique invariant probability measure of maximal entropy for the (semi-)flow on its canonical suspension.*

Moreover, the entropy of the canonical suspension flow is equal to $|\mathcal{A}|$.

1.2.2 Non-degenerating subgraphs

In certain cases, such as those illustrated in Figure 3 and Figure 4, it becomes necessary to consider a subgraph F of a graph G that defines a win-lose induction. Within such subgraphs, we often encounter vertices that have a unique outgoing edge.

From a dynamical point of view, these vertices can be bypassed in the orbit of the win-lose induction, until we encounter a *branching vertex* — that is, a vertex with multiple outgoing edges. This observation motivates the introduction of a *factorization* of the graph.

More precisely, we aim to associate to each *degenerating* subset of labels a corresponding subset of vertices, on which we define an accelerated version of the induction, distinct from the one induced on the factor graph.

To ensure that this acceleration process remains well-defined and finite, we first impose a structural condition on these families. Furthermore, since we wish to keep track of the labels in the degenerating subset, we require that each such label intervenes only once at each step of the accelerated induction.

Definition 1.7 (Filling factoring family)**.** *Consider, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, a subset $\widetilde{V}_{\mathcal{L}}$ of vertices of F such that every loop in this subgraph contains a vertex in $\widetilde{V}_{\mathcal{L}}$. Let $\widetilde{E}^{\mathcal{L}}$ be the set of finite path in F such that*

- *Its start and end vertex belong to $\widetilde{V}_{\mathcal{L}}$ and no other visited vertices do.*
- *Along this path, no letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$.*

The degenerate subgraph $F_{\mathcal{L}}$ is composed of edges appearing in paths of $\widetilde{E}^{\mathcal{L}}$.

We say the collection $\{\widetilde{V}_{\mathcal{L}}\}$ is a filling factoring family *if every $\mathcal{L}$-factor path γ visits at most one branching vertex v satisfying $l(E_v) \cap \mathcal{L} \neq \emptyset$ and which is not the end vertex of γ. We call v the $\mathcal{L}$-branching vertex of γ when it exists.*

Notice that we say a letter α wins against another letter β along a *path* $\gamma = e_1 \dots e_n$ if α wins against β along an edge (in G) or α wins against δ which has won against β before, etc. In other term, if the matrix $M_{e_1} \dots M_{e_n}$ is positive on its coordinate α, β.

A generalization of the non-degenerating criterion is then defined for such families.

Definition 1.8 (Non-degenerating family)**.** *A filling factoring family $\{\widetilde{V}_{\mathcal{L}}\}_{\mathcal{L}}$ forms a* non-degenerating factorization *of the subgraph F if for every $\emptyset \subset \mathcal{L} \subset \mathcal{A}$ and $\mathcal{L}$-factor path $\gamma \in \widetilde{E}^{\mathcal{L}}$ contained in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$ one of these properties is true.*

1. *There exists a path in F starting at a vertex of γ which leaves $\mathscr{C}$ and such that each edge in the path based at a branching vertex of F is labeled in $\mathcal{L}$.*
2. *All edges in γ are labeled within G by letters not in $\mathcal{L}$.*
3. *The path has a $\mathcal{L}$-branching vertex with a unique outgoing edge labeled in $\mathcal{L}$ and each label winning against a letter in $\mathcal{L}$ along γ is in $l(E_v^F) \setminus \{\alpha\}$.*
4. *The path does not meet any $\mathcal{L}$-branching vertex, there is a unique winning label β and at least one losing label not in $\mathcal{L}$.*

Notice in particular similarities of 1 and 3 in this definition with 1 and 2 in Definition 1.6.

A generalization of Theorem A to this setting was also proved in [Fou24].

Theorem A'. *Consider a subgraph of a win-lose induction which has an invariant family of linear subspaces H and a non-degenerating factorization. The map T restricted to $\Delta_H(F, G)$ has a unique invariant measure equivalent to Lebesgue measure and it induces the unique invariant probability measure of maximal entropy for the (semi-)flow on its canonical suspension.*

Moreover, the entropy of the canonical suspension flow is equal to $\dim H$.

1.3 Central Limit Theorem

On $\Delta_H(F,G)$, there is a natural metric given by Hilbert metric on the simplices $\widehat{\Delta}^G$. And one can associate a product metric on $\widehat{\Delta}_r^G$ with the euclidien metric in fibers.

For $\alpha > 0$, let us denote by $H^\alpha(\widehat{\Delta}_r^G)$ the space of α-Hölder functions for this metric. The canonical suspension flow in these cases also satisfies the following.

Theorem B. *Let $p > 2$ and let $\varphi \in H^\alpha(\widehat{\Delta}_r^G) \cap L^p(\widehat{\Delta}_r^G, \mu_r)$ satisfy $\int \varphi \, \mathrm{d}\mu_r = 0$. Assume that there does not exist $\widetilde{\psi} \in L^2(\widehat{\Delta}_r^G, \mu_r)$ differentiable in the direction of the suspension such that $\varphi = X_t \widetilde{\psi}$ where X_t is the Lie derivative in that direction. Then there is a positive constant σ_ϕ such that*

$$\frac{1}{\sqrt{|T|}} \int_0^T \varphi \circ \Phi_t \, dt \xrightarrow{d} \mathcal{N}(0, \sigma_\phi) \quad \textit{as } |T| \to \infty.$$

Where the convergence is in distribution to a normal law of variance σ_ϕ.

2 Rauzy–Veech inductions

2.1 Definitions

For w a finite word in the finite alphabet $\mathcal{A}$ we denote by $|w|_x$ the number of occurrences of the letter $x \in \mathcal{A}$ in the word.

Definition 2.1 (Signed matching)**.** *A signed matching $m = (\nu, \omega, \epsilon)$ is given by two words ν and ω in an alphabet $\mathcal{A}$ which satisfy, for all $x \in \mathcal{A}$,*

$$|\nu|_x + |\omega|_x = 2$$

and a sign map

$$\epsilon : \mathcal{A} \mapsto \{\pm 1\}.$$

For a length vector $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ we denote the length of w by

$$\lambda(w) := \sum_{i=1}^{|w|} \lambda_{w_i}.$$

Definition 2.2 (Linear involution)**.** *Let $m = (\nu, \omega, \epsilon)$ be a signed matching and a length vector $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ such that $\lambda(\nu) = \lambda(\omega)$. We call the couple (m, λ) a linear involution.*

Definition 2.3. *If the sign map of a linear involution (m, λ) satisfies, for all $x \in \mathcal{A}$,*

$$\epsilon(x) = \begin{cases} 1 & \textit{if } |\nu|_x = 1 \\ -1 & \textit{otherwise.} \end{cases}$$

we say that (m, λ) is a linear involution without flips, otherwise with flips.

Definition 2.4 (Interval exchange transformation)**.** *An interval exchange is a linear involution (m, λ) such that for all $x \in \mathcal{A}$, $|\nu|_x = |\omega|_x = 1$. If moreover $\epsilon \equiv 1$ we say that (m, λ) is an interval exchange without flips, otherwise with flips.*

Definition 2.5. *Let $(\nu, \omega, \epsilon, \lambda)$ be a linear involution. We denote $L := \lambda(\nu) = \lambda(\omega)$ and the interval $I = [0, L[$. For each label $l \in \mathcal{A}$, we define two points ξ_l^0, ξ_l^1 in the interval I together with a number $\sigma \in \{0, 1\}$.*

- *If l occurs twice in ν at indices $1 \leq p < q \leq |\nu|$, then:*

$$\xi_l^0 = \sum_{i=1}^{p-1} \lambda_{\nu_i}, \quad \xi_l^1 = \sum_{i=1}^{q-1} \lambda_{\nu_i}, \quad \text{and} \quad \sigma_l = 0.$$

- *If l occurs twice in ω at indices $1 \leq p < q \leq |\omega|$, then:*

$$\xi_l^0 = \sum_{j=1}^{p-1} \lambda_{\omega_j}, \quad \xi_l^1 = \sum_{j=1}^{q-1} \lambda_{\omega_j}, \quad \text{and} \quad \sigma_l = 0.$$

- *If l occurs once in ν at index $1 \leq p \leq |\nu|$ and once in ω at index $1 \leq q \leq |\omega|$, then:*

$$\xi_l^0 = \sum_{i=1}^{p-1} \lambda_{\nu_i}, \quad \xi_l^1 = \sum_{j=1}^{q-1} \lambda_{\omega_j}, \quad \text{and} \quad \sigma_l = 1.$$

For $c \in \mathbb{Z}/2\mathbb{Z}$ and $l \in \mathcal{A}$, we denote by I_l^c the subinterval $]\xi_l^c, \xi_l^c + \lambda_l[\subset I$. Consider the unique linear map $\widetilde{f}_l^c : I_l^c \to I_l^{c+1}$ with constant derivative equal to $\epsilon(l)$. It can be explicitly expressed for all $x \in I_l^c$ by

$$\widetilde{f}_l^c(x) = \xi_l^{c+1} + \epsilon(l) \cdot \left(x - \xi_l^c - \frac{\lambda_l}{2}\right) + \frac{\lambda_l}{2}.$$

Let S be the set of all points ξ_l^c. We define an involution $\widetilde{f}$ on $I \setminus S \times \mathbb{Z}/2\mathbb{Z}$ by, for $x \in I_l^c$ and $\sigma_0 \in \mathbb{Z}/2\mathbb{Z}$,

$$\widetilde{f}(x, \sigma_0) = \left(\widetilde{f}_l^c(x), \sigma_0 + \sigma_l\right).$$

Remark 2.6. *This associated map motivates the name for the couple of signed matching and length vector. They correspond to linear involutions with and without flips, as considered in [DN90] and [BL09] respectively. In the case of interval exchange, the map always changes the element in $\mathbb{Z}/2\mathbb{Z}$ and can be factored into a translation map on the interval I, as defined in [Yoc10].*

Moreover, this association can clearly be performed in the other direction, from the involution map to a signed matching with a length vector.

The existence of a linear involution is central in the definition of Rauzy–Veech induction. But for some signed matching the condition on lengths can clearly not be met. For instance, see the matching with all double letters on top in Figure 3.

Definition 2.7. *We say a signed matching $m = (\nu, \omega, \epsilon)$ is balanced if m is an interval exchange or there exists a letter a in ν and b in ω such that $|\nu|_a = |\omega|_b = 2$.*

Proposition 2.8. *A signed matching m is balanced if and only if there exists a length parameter λ such that (m, λ) is a linear involution.*

Proof. Assume without loss of generality that all letter α in ν are such that $|\nu|_\alpha = 1$, then $\lambda(\omega) - \lambda(\nu) = 2\#\{\lambda_\beta \mid \beta \in \mathcal{A} \text{ and } |\omega|_\beta = 2\} = 0$. Which can only be true if the set of double letters is empty.

Conversely, the existence of a length vector is straightforward for interval exchanges and when there are double intervals on top and bottom. □

We denote by $\Sigma(\mathcal{G}_n)$ the set of signed matching on n letters and $\Sigma_0(\mathcal{G}_n)$ the subset of signed matching such that either it is unbalanced or the last letters of the words ν and ω are equal. For $x, y \in \mathcal{A}$, let us introduce the substitutions

$$s_{x,y}^1 : x \to x \cdot y$$

and

$$s_{x,y}^{-1} : x \to y \cdot x.$$

For all non-empty word w we denote by $\widetilde{w}$ the same word to which we have removed the last letter.

Definition 2.9. *The Rauzy–Veech induction for $n \geq 2$ is the map*

$$\begin{aligned} \mathcal{R}_n : \quad \Sigma(\mathcal{G}_n) \setminus \Sigma_0(\mathcal{G}_n) \times \mathbb{R}_+^n &\longrightarrow \Sigma(\mathcal{G}_n) \times \mathbb{R}_+^n \\ (\nu, \omega, \epsilon, \lambda) &\longmapsto (\nu', \omega', \epsilon', \lambda'). \end{aligned}$$

where for α, β the (distinct) last letters of ν and ω, the image is defined as follows :

- *If $\lambda_\alpha > \lambda_\beta$,*

$$\nu' = s_{\alpha,\beta}^{\epsilon(\alpha)}(\widetilde{\nu}) \cdot \alpha,$$
$$\omega' = s_{\alpha,\beta}^{\epsilon(\alpha)}(\widetilde{\omega}),$$

$$\epsilon'(\beta) = \epsilon(\alpha) \cdot \epsilon(\beta),$$
$$\lambda'_\alpha = \lambda_\alpha - \lambda_\beta.$$

- *If $\lambda_\beta > \lambda_\alpha$,*

$$\nu' = s_{\beta,\alpha}^{\epsilon(\beta)}(\widetilde{\nu}),$$
$$\omega' = s_{\beta,\alpha}^{\epsilon(\beta)}(\widetilde{\omega}) \cdot \beta,$$

$$\epsilon'(\alpha) = \epsilon(\beta) \cdot \epsilon(\alpha),$$
$$\lambda'_\beta = \lambda_\beta - \lambda_\alpha.$$

The coordinates that are not mentioned for ϵ' and λ' are kept unchanged.

Invariant linear form The difference $\lambda(\nu) - \lambda(\omega)$ is preserved by the Rauzy–Veech induction. Classically, the Rauzy–Veech induction is only defined in the case $\lambda(\nu) = \lambda(\omega)$ on maps associated to linear involution or interval exchanges.

In other terms, the family linear forms defined to each vertex (ν, ω, ϵ) of $\mathcal{G}_n$ defined for all $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ by

$$\delta_{(\nu,\omega,\epsilon)}(\lambda) = \lambda(\nu) - \lambda(\omega)$$

is preserved by composition with the induction. Hence its kernels form an invariant family of linear subspaces.

Proposition 2.10. *The linear form δ_m intersects the positive cone if and only if m is balanced. In particular, for m to intersect the positive cone it is necessary that $m \in \mathcal{F}_n$.*

Geometric interpretation Let us mention here that there is a geometric interpretation of these maps. It is not necessary to our definition but may help the reader to understand its intuition. As in Section 2.1 of [BL09], the linear involution can be seen as the first return map for a foliation on a surfaces on a transverse interval. The interval is duplicated to separate cases where the leaf arrives at the top or bottom of the interval.

Let s is the map switching values 0 and 1 in the second coordinate, *i.e.* for $(x, \sigma_0) \in I \setminus S \times \mathbb{Z}/2\mathbb{Z}$ by $s(x, \sigma_0) = (x, \sigma_0 + 1)$, the orbits of the composed map $s \circ \widetilde{f}$ correspond to the intersection of the leaves of the foliation with the interval. Such maps associated to a foliation depend on the choice of interval and Rauzy–Veech induction is a natural induction which builds up from a linear involution another one implied by the first return map of the same foliation on a different interval.

The following is proved in Section 2.2 of [BL09].

Proposition 2.11. *Let $L = (v, w, \epsilon, \lambda)$ be a linear involution. The linear involution L' is the image by Rauzy–Veech induction of L if and only if its associated map $s \circ \widetilde{f}_{L'}$ is the first return map of $s \circ \widetilde{f}_L$ on*

$$]0, \max\left(\lambda(\widetilde{\nu}), \lambda(\widetilde{\omega})\right)[\, \times \mathbb{Z}/2\mathbb{Z}.$$

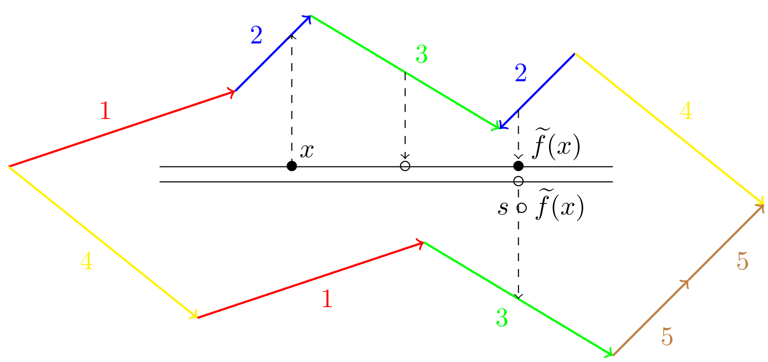


Figure 5: Linear involution as a first return map of a vertical foliation.

2.2 Win-lose induction

Rauzy–Veech induction can be seen as the win-lose map associated to graph $\mathcal{G}_n$ whose vertices are all the signed matching in $\Sigma(\mathcal{G}_n)$. Edges going out of a vertex associated to a given signed matching $(v, w, \epsilon) \in \Sigma(\mathcal{G}_n) \setminus \Sigma_0(\mathcal{G}_n)$ are defined in Figure 6. The change on ϵ maps are not written down to simplify the presentation but are clear from the definition of the induction given above.

The image signed matching by Rauzy–Veech induction can be in $\Sigma_0(\mathcal{G}_n)$. In this case, the Rauzy–Veech induction is not defined and stops. Such vertices are thus defined in $\mathcal{G}_n$ to have no outgoing edges.

Property 2.12. *For a vertex v, if there is a letter in the labels of E_v that is not in $\mathcal{L}$, this property is preserved along $\mathcal{L}$-factor paths in $\mathcal{G}_n$ or a subgraph of it.*

$$\begin{pmatrix} s_{\alpha,\beta}^{\epsilon(\alpha)}(\widetilde{\nu})\cdot\alpha \\ s_{\alpha,\beta}^{\epsilon(\alpha)}(\widetilde{\omega}) \end{pmatrix} \xleftarrow{\quad\beta\quad} \begin{pmatrix} \nu \\ \omega \end{pmatrix} \xrightarrow{\quad\alpha\quad} \begin{pmatrix} s_{\beta,\alpha}^{\epsilon(\beta)}(\widetilde{\nu}) \\ s_{\beta,\alpha}^{\epsilon(\beta)}(\widetilde{\omega})\cdot\beta \end{pmatrix}$$

Figure 6: Outgoing edges for a given signed matching

Proof. It is a key property of Rauzy–Veech induction that a letter winning along an edge from v to v' appears in the labels of edges going out of the ending vertex v'. But by the definition of $\mathcal{L}$-factor path of a subgraph, there is no letter in which a letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$. Thus, as there is at least one letter not in $\mathcal{L}$ going out of v, no matter whether the losing letter is in $\mathcal{L}$ or not, the winning letter is not. And this is true all along the branch path. □

Property 2.13. *Consider a path γ starting with an edge e from vertex v. In $\mathcal{G}_n$, if α wins against e, then either it is contained in $l(E_{\gamma\cdot v})$ or a letter in this latter set wins against α along γ.*

Proof. Again we use the fact that the winning label remains in the outgoing edge of the next vertex. Then notice that if β wins against α along γ_1 and δ wins against β along the next edge e' then δ wins against α along $\gamma_1 \cdot e'$ and δ appears in labels of the edges going out of the ending vertex of e'. We then prove the proposition by induction. □

Let $\mathcal{F}_n$ be the subgraph of $\mathcal{G}_n$ from which we remove vertices in $\Sigma_0(\mathcal{G}_n)$ and edges pointing to them. We denote them respectively by F and G.

A strongly connected component of a directed graph is by definition a maximal subgraph such that all distinct vertices v and v' have a directed path from v to v' and from v' to v.

We can associate to any directed graph a directed acyclic graph which vertices are labeled by strongly connected components and for which we draw an edge between two vertices if there is an edge connecting the two strongly connected components in the graph. It is classically called the condensation graph of a directed graph (see for instance section 3.4 of [BM08]). There are *minimal vertices* in this acyclic graph which have no outgoing edges. Vertices in the corresponding strongly connected components, called the *minimal component*, have no edges pointing to another strongly connected component in G. In particular the only outgoing edges that appear in $\mathcal{G}_n$ and not in such a component are pointing to vertices in $\Sigma_0(\mathcal{G}_n)$.

Definition 2.14. *A strongly connected components of $\mathcal{F}_n$ for which the set of labels of edges is the whole alphabet $\mathcal{A}$ is called an* irreducible component*. A signed matching in such a component is also called* irreducible.

For interval exchanges, irreducible signed matching (ν, ω, ϵ) is fully characterized by the fact that their is no non-trivial decomposition $\nu = \nu^1 \cdot \nu^2$ and $\omega = \omega^1 \cdot \omega^2$ such that the set of labels appearing in ν^1 and ω^1 (and thus in ν^2 and ω^2) are equal (see *e.g.* [Yoc10]).

For linear involutions without flips, irreducible signed matching are characterized in [BL09] Definition 3.1. It is also proved in that work that such objects come from a geometric model (as the first return map on an interval of the vertical foliation of a half-translation surfaces).

It would be interesting to consider the following question.

Question 2.15. *Characterize irreducible signed matchings in cases with flips.*

Property 2.16. *Let v be a vertex in an irreducible component such that there is a unique edge labeled by α going out of a v. Let γ be the path in $\mathcal{G}_n$ from v to v' such that only its start and end vertex have an outgoing edge labeled by α in $\mathcal{G}_n$. Then, either v' has two outgoing edges or if β is the letter that loses against α at the first edge, v' has a unique outgoing edges and α wins against its label along this edge.*

Proof. Let us consider a path from $\widehat{F}^{\mathcal{L}}$ to itself. Assume it contains a loop γ starting (and ending) at a branching vertex for which none of the labels of outgoing edges are in $\mathcal{L}$. Let $e : v \to v'$ be the first edge in γ such that there is an edge going out of v' which is labeled in $\mathcal{L}$. Let α be the label of e and β the label winning against α along e. For all label δ winning against a label in $\mathcal{L}$ at an edge in γ we have either $\beta = \delta$ or δ wins against β along γ. □

Proposition 2.17. *Irreducible components of the Rauzy diagram are non-degenerating.*

Let us define

$$\widetilde{V}_{\mathcal{L}} = \left\{ v \in G_{\mathcal{L}} \mid E_v^G \cap \overline{\mathcal{L}} = \emptyset \text{ or } E_v^F \cap \mathcal{L} = \emptyset \text{ or } |E_v^F| = 2 \right\}.$$

Proposition 2.18. *A $\mathcal{L}$-factor path associated to $\{\widetilde{V}_{\mathcal{L}}\}$ cannot loop.*

Proof. Assume there is a loop that does not contain a vertex in $\widetilde{V}_{\mathcal{L}}$. Then it is composed of non branching vertices labeled in F and its edge are labeled in $\overline{\mathcal{L}}$. The loop then composes the whole strongly connected component and is labeled in a strict subset of the alphabet. Which contradicts the irreducibility property. □

Proof of Proposition 2.17. Let γ be a $\mathcal{L}$-factor path starting at a vertex v and contained in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$.

If $E_v^G \cap \overline{\mathcal{L}} = \emptyset$. As F is strongly connected and contains all labels, there exists a path γ starting at v in F which contains a vertex with a least one label that is not in $\mathcal{L}$. Up to taking a prefix of γ, we assume its end vertex is the first vertex to satisfy this condition. Thus all previous branching vertices have both their outgoing edges labeled in $\mathcal{L}$. By Property 2.12 the path γ must leaves the strongly connected component $\mathscr{C}$ of v. Condition 1 is then satisfied.

If $E_v^F \cap \mathcal{L} = \emptyset$. Let $\alpha \in \overline{\mathcal{L}}$ be the first label γ and β the label of the other edge in G. Then β must also be in $\overline{\mathcal{L}}$ since otherwise a letter in $\mathcal{L}$ would win against a letter in $\overline{\mathcal{L}}$ along γ. Again by Property 2.12, in a $\mathcal{L}$-factor path starting at v, all vertices have at least one outgoing edge not labeled in $\mathcal{L}$. Its ending vertex in $\widetilde{V}_{\mathcal{L}}$ then must satisfy $E_v^F \cap \mathcal{L} = \emptyset$. If the path is composed of only one edge, this falls in condition 2.

Otherwise, in intermediate steps there is always one edge labeled in $\mathcal{L}$ in F which must lose. And the winning letter along each edge of the path is always β since is must be preserved by the fundamental property of Rauzy–Veech induction. Thus condition 4 is satisfied.

If $|E_v^F| = 2$. Assuming the other condition are false, v is a branching vertex with an outgoing edge $\alpha \in \mathcal{L}$ and $\beta \in \overline{\mathcal{L}}$. Again, along every edges of γ, the winning letter must be β. If next vertex does not have an outgoing edge in $\mathcal{L}$, we have condition 2. We only have to consider the case when the path continues with edges labeled in $\mathcal{L}$ from a vertex having a unique outgoing edge in F. In this case, the letter winning along the first edge, also wins along the next edges labeled in $\mathcal{L}$. Satisfying condition 3. □

2.3 Consequences for IET and LI with flips

We first mention the following proposition.

Proposition 2.19. *For all IET or LI with flips, there is a path to a signed matching have the same two ending labels.*

As a consequence, we can bound the Hausdorff dimension of IET and LI with flips which are not minimal.

Theorem 2.20. *For a given signed matching m corresponding to a IET with flips, the set of lengths $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ such that a (m, λ) does not contain a periodic orbit is strictly smaller than $|\mathcal{A}|$.*

For LI, one needs to consider the restriction of the induction to the set of length in the kernel of δ.

Theorem 2.21. *For a given signed matching m corresponding to a LI with flips, the set of lengths $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ such that a (m, λ) does not contain a periodic orbit is strictly smaller than $|\mathcal{A}| - 1$.*

As noticed in Proposition 2.10, one can induced the win-lose map on the subgraph of $\mathcal{G}_n$ on balanced signed matching by considering parameters in the kernel of the invariant family of linear forms δ.

3 Natural extensions

3.1 Zippered rectangles

In the case of interval exchange transformation a geometrical parametrization of the natural extension of Rauzy–Veech induction has been introduced by Veech in [Vee78]. He has named this construction zippered rectangles, expressing in a visual way the intuition behind it. This construction was generalized to linear involution in [BL09].

Assume ν, ω and ϵ define a signed matching associated to an interval exchange of linear involution (hence ϵ is fully determined by the words). Assume there is a (half-)translation surface that suspends the associated map or in other words that one can find n vectors ν in $\mathbb{C}$ labeled in $\mathcal{A}$ such that

- for all $\alpha \in \mathcal{A}$
$$\Re(\nu_\alpha) > 0,$$
- for all $1 < k < |\nu|$,
$$\sum_{\alpha \in \nu(1\ldots k)} \Im(\nu_\alpha) > 0,$$
- for all $1 < k < |\omega|$,
$$\sum_{\alpha \in \omega(1\ldots k)} \Im(\omega_\alpha) < 0.$$

Where $w(1 \ldots k)$ denotes the length k prefix of the word w.

Let us now consider the polygon obtained by representing these vectors starting at point 0 one after the other in both orderings given by ν and ω. We identify pairs of vectors with matching labels by translation or translation composed with central symmetry (when ϵ is respectively positive or negative). Then the obtained translation surface suspends the linear involution defined by the given signed matching and the lengths $(\Re(v_\alpha))_{\alpha \in \mathcal{A}}$.

On this surface, one can represent the suspension data by considering the first return of the vertical flow on the horizontal interval. Figure 7 represents these two constructions.

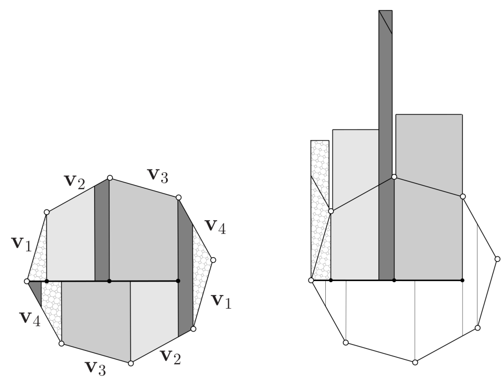


Figure 7: Zippered rectangle construction.

Let $\mathcal{A}_m$ be a marked alphabet containing two copies x_0, x_1 of each letter $x \in \mathcal{A}$. We assume moreover, up to choosing a marking, that ν and ω are words in $\mathcal{A}_m$ such that each copy of a letter appears exactly once in the concatenation of the words.

Let σ be the involution of $\{0,1\}$ which switches the two elements. For every $x \in \mathcal{A}$, let f_x be the bijections on $\{0,1\}$ defined as

$$f_x = \begin{cases} id & \text{if } \epsilon(x) = 1, \\ \sigma & \text{otherwise} \end{cases} .$$

We will use the notation $\iota_x(ij) := f_x(i) f_x(j)$.

For each interval, we consider the rectangle formed by the suspension of the interval above the top and below the bottom domains corresponding to ν and ω. The corresponding side in the polygonal surface representation cuts it horizontally in a top and bottom half. Each vertical side are cut into two pieces — except for the rectangles meeting the leftmost singularity. We denote their length by Z_{ij} where i, j are 0 or 1 for respectively left, right and down, up interval.

In the wording of Veech, these numbers define *zip heights* which express the position of the singularities and enable us to reconstruct the surface. These heights satisfy the properties in the following definition.

Definition 3.1 (Zip functions). *Let m be a signed matching of linear involution. Let $\alpha_k, \beta_l \in \mathcal{A}$ be the last letters of ν and ω respectively and γ_m, η_n the first letters. Zip functions for m are functions $Z_{00}, Z_{01}, Z_{10}, Z_{11} : \mathcal{A}_m \to \mathbb{R}$ such that*

1. $Z_{00} + Z_{01} = Z_{10} + Z_{11}$,

2. $Z_{ij}(x_{\sigma(k)}) = Z_{\iota_x(ij)}(x_k)$ *for all* $x_k \in \mathcal{A}_m$ *and* $i, j \in \{0, 1\}$,
3. $Z_{1i}(x_k) = Z_{0i}(y_l)$ *for all* $x_k, y_l \in \mathcal{A}_m$ *such that* $x_k y_l$ *is a factor of* w_i,
4. $Z_{\iota_\alpha(10)}(\alpha_{\sigma(k)}) = Z_{10}(\alpha_k) = -Z_{11}(\beta_l) = -Z_{\iota_\beta(11)}(\beta_{\sigma(l)})$,

$$Z_{00}(\gamma_m) = Z_{\iota_\gamma(00)}(\gamma_{\sigma(m)}) = 0,$$

$$Z_{01}(\eta_n) = Z_{\iota_\eta(01)}(\eta_{\sigma(n)}) = 0.$$

5. *for all* $x_k \in \alpha_m$ *and* $i, j \in \{0, 1\}$ *such that* $Z_{ij}(x_k)$ *does not appear in the previous list,*

$$Z_{ij}(x_k) > 0.$$

Condition 2 can be understood according to Figure 8.

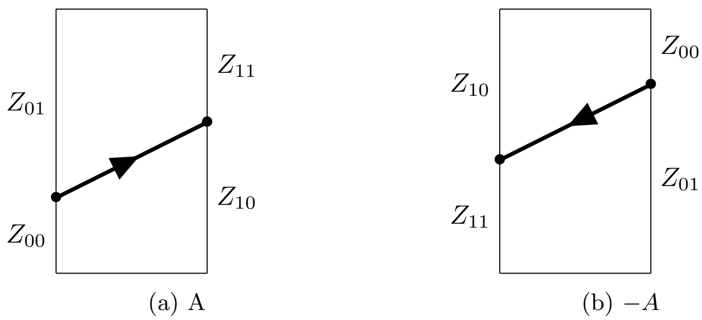


Figure 8: Identification of the zip functions when applying -*id*.

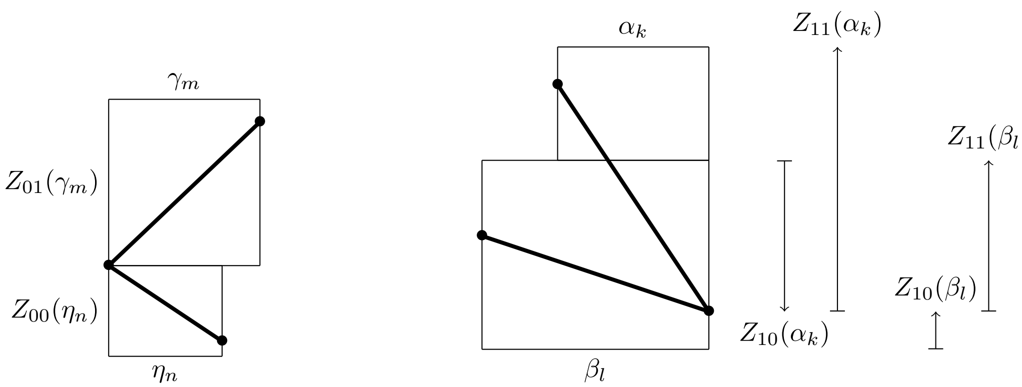


Figure 9: Zip functions around the leftmost and rightmost singularities.

These functions induce a height map $h : \mathcal{A} \to \mathbb{R}_+$ defined for all $\alpha \in \mathcal{A}$ and $i, j \in \{0, 1\}$,

$$h(\alpha) := Z_{i0}(\alpha_j) + Z_{i1}(\alpha_j).$$

According to properties 1 and 2 of the definition, this does not depend on i, j.

Remark 3.2. *We denote by $\mathcal{Z}_m$ the space of maps satisfying these conditions for a signed matching m. Notice that it is a subcone of a vector space on which all coordinates but one (corresponding to Condition 4) are assumed to be positive.*

Let

$$\mathcal{S}_n := \bigsqcup_{m \in \Sigma(\mathcal{G}_n) \setminus \Sigma_0(\mathcal{G}_n)} (\{m\} \times \mathbb{R}_+^n \times \mathcal{Z}_m)$$

be an extension of the parameter space $\Sigma(\mathcal{G}_n) \setminus \Sigma_0(\mathcal{G}_n) \times \mathbb{R}_+^n$ on which Rauzy–Veech induction is defined. One can extend Rauzy–Veech induction

$$\widehat{\mathcal{R}}_n : (\nu, \omega, \epsilon, \lambda, Z) \in \mathcal{S}_n \longmapsto (\nu', \omega', \epsilon', \lambda', Z') \in \mathcal{S}_n$$

Where $(\nu', \omega', \epsilon')$ and λ' are like in Definition 2.9 and Z' is defined by the formulas below.

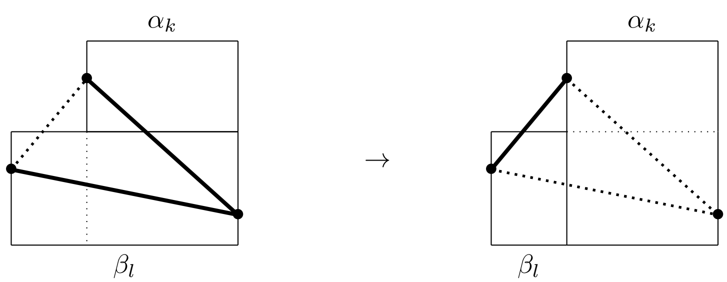
These formulas are motivated by an induction on the underlying surface of a zippered rectangles construction. We cut off a triangle on the left-hand side of the surface and glue one side of the triangle to the polygon.

Remark 3.3. *We express Rauzy–Veech induction on only one marking, the action on the other marking of the same letter is defined to preserve Property 2 in the definition. Other values that do not appear in the definition are unchanged.*

Formulas for induced zip functions.

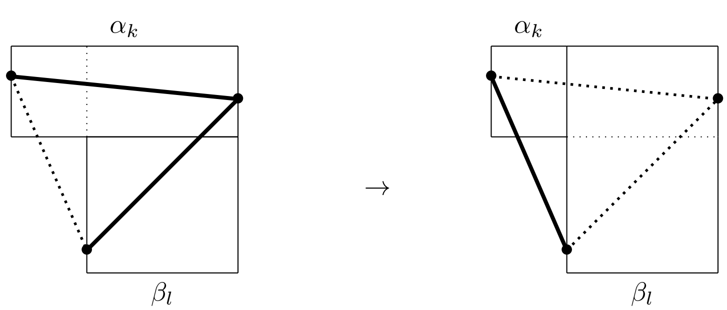
If bottom is winning,

$$
\begin{aligned}
Z'_{10}(\beta_l) &= Z_{00}(\alpha_k) && +h(\beta_l) \\
Z'_{11}(\beta_l) &= -Z_{00}(\alpha_k) \\
Z'_{\iota_\beta(00)}(\alpha_k) &= Z_{00}(\alpha_k) && +h(\beta_l) \\
Z'_{\iota_\beta(01)}(\alpha_k) &= Z_{01}(\alpha_k) \\
Z'_{\iota_\beta(10)}(\alpha_k) &= Z_{10}(\alpha_k) && +h(\beta_l) && = Z_{10}(\beta_l) \\
Z'_{\iota_\beta(11)}(\alpha_k) &= Z_{11}(\alpha_k) && && = Z_{11}(\beta_l) + h(\alpha_k)
\end{aligned}
$$



If top is winning,

$$
\begin{aligned}
Z'_{10}(\alpha_k) &= -Z_{01}(\beta_l) \\
Z'_{11}(\alpha_k) &= Z_{01}(\beta_l) && +h(\alpha_k) \\
Z'_{\iota_\alpha(00)}(\beta_l) &= Z_{00}(\beta_l) \\
Z'_{\iota_\alpha(01)}(\beta_l) &= Z_{01}(\beta_l) && +h(\alpha_k) \\
Z'_{\iota_\alpha(10)}(\beta_l) &= Z_{10}(\beta_l) && && = Z_{10}(\alpha_k) + h(\beta_l) \\
Z'_{\iota_\alpha(11)}(\beta_l) &= Z_{11}(\beta_l) && +h(\alpha_k) && = Z_{11}(\alpha_k)
\end{aligned}
$$



Proposition 3.4. *If m' is the image signed maching then $Z' \in \mathcal{Z}_{m'}$.*

Proof. This fact is clear when considering the geometric interpretation of the induction. But we will check it directly on the formulas to convince the reader that they correspond to the construction.

Let us consider the case where bottom — letter β — is winning. The other case is closely similar.

1. Property 1 in the definition is obviously preserved.
2. Property 2 is preserved since we defined the induction on the other marking using this property.
3. Two consecutive letters $x_r y_s$ remain consecutive in ν' and ω' if $x_r \neq \beta_{\sigma(l)}$ and $y_s \neq \alpha_k$.

   If $\epsilon(\beta) = 1$, a factor $\beta_{\sigma(l)} y_s$ appears in ν' and we have two new consecutive couple of letters $\beta_{\sigma(l)}\alpha_k$ and $\alpha_k y_s$ after induction. Remark from the equations that $Z'_{10}(\beta_l) = Z'_{\iota_\beta(00)}(\alpha_k) = Z'_{00}(\alpha_k)$ hence $Z'_{10}(\beta_{\sigma(l)}) = Z'_{10}(\beta_{\sigma(l)}) = Z'_{00}(\alpha_k)$. And $Z'_{10}(\alpha_k) = Z_{10}(\beta_l) = Z'_{00}(y_s)$.

   If $\epsilon(\beta) = -1$, a factor $x_r\beta_{\sigma(l)}$ appears in ω' and we have at most two new consecutive couple of letters $x_r\alpha_k$ and $\alpha_k\beta_{\sigma(l)}$ after induction. Again we have $Z'_{10}(\beta_l) = Z'_{\iota_\beta(00)}(\alpha_k)$ hence $Z'_{11}(\alpha_k) = Z'_{01}(\beta_{\sigma(l)})$. And $Z'_{11}(x_r) = Z_{01}(\beta_{\sigma(l)}) = Z_{10}(\beta_l) = Z'_{01}(\alpha_k)$.
4. Let $\alpha'_{k'}$ be the last letter of ν' and $\eta'_{n'}$ the first letters of ω'. If $\alpha' \neq \alpha$ then $\alpha'_{k'}\alpha_k$ is a factor of ν and

   $$Z'_{10}(\alpha'_{k'}) = Z_{10}(\alpha'_{k'}) = Z_{00}(\alpha_k) = -Z'_{11}(\beta_l).$$

   If $\epsilon(\beta) = 1$ the first letters of ν and ω remain unchanged. But we can have $\alpha' = \alpha$. In this case, $\beta_{\sigma(l)}\alpha_k$ is a factor of ν' and

   $$Z'_{10}(\alpha'_{k'}) = Z'_{\iota_\beta(10)}(\alpha_k) = Z_{10}(\beta_l) = Z_{10}(\beta_{\sigma(l)}) = Z_{00}(\alpha_k) = -Z'_{11}(\beta_l).$$

   If $\epsilon(\beta) = -1$, the unique delicate case is when $\beta_{\sigma(l)}$ is the first letter of ω. Then $\eta'_{n'} = \alpha_k$ and $\alpha_k\beta_{\sigma(l)}$ is a factor of ω'. Thus

   $$Z'_{01}(\eta'_{n'}) = Z'_{\iota_\beta(10)}(\alpha_k) = Z_{10}(\beta_l) = Z_{01}(\beta_{\sigma(l)}) = 0$$
5. Finally $Z'_{\iota_\beta(ij)(\alpha_k)}$ becomes positive since we add heights to potentially negative zip values. Only $Z'_{\iota_\beta(01)}(\alpha_k) = Z_{01}(\alpha_k)$ may be zero but this would mean that either $\epsilon(\alpha) = 1$ and $\alpha_{\sigma(k)} = \eta_n$ or $\epsilon(\alpha) = -1$ and $\alpha_{\sigma(k)} = \gamma_m$ which is not possible in a linear involution.

□

By the interpretation of zippered rectangles as suspension flow of the linear involution and Rauzy–Veech induction as a first return map, the suspension time above the losing interval should be added the one above the winning interval. We again prove this property using the defining formulas.

Proposition 3.5. *Let h and h' in $\mathbb{R}^{\mathcal{A}}$ be the height function and its image by the extended Rauzy–Veech induction. If M_e is the matrix of the induction defined in the win-lose formalism, then*

$$h' = M_e^t h.$$

Proof. Just notice that, in the formulas, the height of the winning letter is unchanged and we had the height of the winning letter to the height of the losing one. ☐

Definition 3.6. *Let*

$$\Delta_m^0 := \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha < \lambda_\beta\}$$

and

$$\Delta_m^1 := \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_\alpha > \lambda_\beta\}$$

where α, β denote the last letters of the two words of m. These length parameter spaces correspond to bottom or top winning respectively.

Let

$$\mathcal{Z}_m^0 := \{Z \in \mathcal{Z}_m \mid Z_{10}(\alpha_k) > 0\} = \{Z \in \mathcal{Z}_m \mid Z_{11}(\beta_l) < 0\}$$

and

$$\mathcal{Z}_m^1 := \{Z \in \mathcal{Z}_m \mid Z_{10}(\alpha_k) < 0\} = \{Z \in \mathcal{Z}_m \mid Z_{11}(\beta_l) > 0\}$$

where α_k, β_l denote the last letters of the words of m in the marked alphabet.

Proposition 3.7. *For any signed matching m, Rauzy–Veech induction restricted to $\big(\{m\} \times \Delta_m^0 \times \mathcal{Z}_m\big) \sqcup \big(\{m\} \times \Delta_m^1 \times \mathcal{Z}_m\big)$ defines two invertible linear maps*

$$R_0(m) : \{m\} \times \Delta_m^0 \times \mathcal{Z}_m \longrightarrow \{m_0\} \times \Delta_{m_0} \times \mathcal{Z}_{m_0}^0$$

and

$$R_1(m) : \{m\} \times \Delta_m^1 \times \mathcal{Z}_m \longrightarrow \{m_1\} \times \Delta_{m_1} \times \mathcal{Z}_{m_1}^1.$$

Proof. Let us consider the case where bottom is winning. Injectivity is obvious from the definition thus we only need to prove that the image of zip functions is $\mathcal{Z}_{m_0}^0$. The inclusion in $\mathcal{Z}_{m_0}$ was proved in the previous proposition, let us show that $Z'_{11}(\beta_l) < 0$. Notice that by definition we have $Z'_{11}(\beta_l) = -Z_{00}(\alpha_k) = -Z_{\iota_\alpha(00)}(\alpha_{\sigma(k)})$ and $\alpha \neq \beta$. If α does not appear as a letter at the beginning of ν or ω then condition 5 implies the inequality its values clearly cover the whole set. If α is the first letter of ν, then $\epsilon(\alpha) = -1$ and $Z_{00}(\alpha_{\sigma(k)}) = Z_{11}(\alpha_k) = 0$. If α is the first letter of ω, then $\epsilon(\alpha) = 1$ and $Z_{01}(\alpha_{\sigma(k)}) = Z_{01}(\alpha_k) = 0$. In these two cases, the equality does not affect the value of $Z_{00}(\alpha_k)$ which is then again positive by condition 5.

Finally, negative zip heigths become positive by the last part of the previous proof and their values again cover all positive values. ☐

Proposition 3.8. *The extended Rauzy map $\widehat{\mathcal{R}}_n$ is a bijection on its image.*

Proof. Assume bottom has won, one can revert the action on the sign matching by finding the losing letter. It must be next to the twin label of the winner. The value of ϵ being unchanged for the winner (which is the last letter on top). And it is the same if top has won.

One can then construct the inverse of $\widehat{\mathcal{R}}_n$. We decide which letter has won by looking at the sign of $Z_{11}(\beta_l)$ and take the inverse by the corresponding map in the previous proposition. ☐

Consider H_m the vector subspace of $\mathbb{R}^{\mathcal{A}_m}$ satisfying equalities in conditions 1–4. Consider the canonical basis of $\mathbb{R}^{\mathcal{A}_m}$ and extract a basis for H_m which be endowed with a canonical scalar product. The space $\mathcal{Z}_m$ is a subcone of H_m where all but one coordinates in the basis are positive.

From the definition equations we have the following decomposition of the linear map associated to the extended Rauzy–Veech induction.

Proposition 3.9. *Let m be a signed matching and m' its image throught R_i with $i \in \{0,1\}$. There exists an orthogonal map $U : H_m \mapsto H_{m'}$ and a vector $v(h) \in H_m$, whose coordinates are positive linear combinations of heights, such that $\widehat{\mathcal{R}}_n$ acts on H_m as $U + v(h)$.*

In particular, for a given zip fonction Z, the image zip function $Z^{(k)}$ after k steps of induction, can be expressed as $Z^{(k)} = U^{(k)} Z + v^{(k)}$ where $U^{(k)}$ is an orthogonal matrix and $v^{(k)}$ is a linear combination of heights along the path $h(0), h(1), \ldots, h(k-1)$.

By Proposition 3.5, the scalar product between λ and h is preserved. It correspond geometrically to the area of the surface. One can thus consider the induction restricted to the subspace of parameters where the scalar product $\langle \lambda, h \rangle = 1$. On this subspace we have the following key lemma.

Lemma 3.10. *The extended Rauzy–Veech induction restricted to parameters where $\langle \lambda, h \rangle = 1$ is the natural extension of Rauzy–Veech induction.*

Proof. We proved in Proposition 3.8 that the extended Rauzy–Veech induction is a bijection. The set of parameters with $\langle \lambda, h \rangle = 1$ projects surjectively to all possible lengths. It remains to prove that the coding in the past of a given zip function Z determines it uniquely for almost all paths. We follow Bufetov's scheme of proof in [Buf06].

Consider a path $\gamma_* = e_1 \ldots e_n$ in the Rauzy diagram for which the associated path matrix $A = M_{e_1} \ldots M_{e_n}$ is positive. Let us consider the unique invariant measure equivalent to Lebesgue. For almost every lengths and zip parameters, in the corresponding bi-infinite path

$$\gamma = \ldots \gamma_{-k} \gamma_{-k+1} \ldots \gamma_{-1} \gamma_0 \gamma_1 \ldots \gamma_{k-1} \gamma_k \ldots$$

in the Rauzy diagram, the positive path γ_* appears infinitely many times in the future and the past of the coding (see e.g. Proposition 2.8 in [Fou24]). Moreover, by Proposition 3.5, the action of Rauzy–Veech induction on the heights of zippered rectangles is given by the transpose of paths matrices. Thus if $M_{(-k)}$ is the matrix induced by win-lose induction along $\gamma_{-k} \ldots \gamma_0$, $h \in M^t_{(-k)} \cdot \mathbb{R}^{\mathcal{A}}_+$. As A is uniformly contracting Hilbert distance on the projectivized parameter space,

$$\bigcap_{k=0}^{\infty} M^t_{(-k)} \cdot \mathbb{R}^{\mathcal{A}}_+ = \mathbb{R}_+ \cdot h_\infty$$

is reduced to one ray and thus the heights h is defined uniquely up to a multiplicative constant by the past coding $\ldots \gamma_{-k} \gamma_{-k+1} \ldots \gamma_0$ (see e.g. section 3.1 in [Fou24]). The condition on scalar product then defines it uniquely.

Moreover, coefficients of $M_{(-k)}$ go to infinity as k goes to infinity. The height of zippered rectangles at step $-k$ satisfies $M^t_{(-k)} h(-k) = h$ thus $h(-k) \to 0$ and $Z^{(-k)} \to 0$ since its coefficients are bounded by the heights.

Using the previous proposition, we see that the zip functions $Z^{(-k)}$ such that after k steps of Rauzy–Veech induction we get that Z can be expressed as

$$Z = U^{(-k)} Z^{(-k)} + v^{(-k)}$$

with $U^{(-k)}$ orthogonal and $v(-k)$ depending only on heigths $h(-k), h(-k+1), \ldots, h(-1)$ which themselves are determined by the past coding. Hence $Z = \lim_{n\to\infty} v^{(-n)}\,(h(-n))$ and thus Z only depends on the past coding for almost every points. □

Remark 3.11. *The Lebesgue measure on the space of zippered rectangles is clearly invariant by extended Rauzy–Veech induction. This measure can be introduced intrinsically directly on strata of Abelian or quadratic differentials and is commonly called Masur–Veech measure (see [Zor06] for more background).*

3.2 Teichmüller flow

By classical results of the theory, Teichmüller flow is finite-to-one semi-conjugated to the extension of Rauzy–Veech induction to zippered rectangles. According to Lemma 3.10, it corresponds to natural extension of Rauzy map.

Thus, the ergodic ergodic properties of the canonical suspension flow for Rauzy–Veech induction ergodic properties for the Teichmüller flow on connected components of strata of abelian or quadratic differentials.

In the following we show that Masur–Veech measure is the unique measure of maximal entropy for these flows as well as Central Limit Theorem. These results where known in the case of Abelian differentials [BG11], [Buf06], . This also implies a common proof for exponential mixing which where proved by [AGY06] in the case of Abelian differential and [AR12] for quadratic differentials.

Abelian differentials Let $\mathcal{R}$ be a connected component of Rauzy diagram. Let $\widetilde{T} : \widetilde{\Delta}_{\mathcal{R}} \to \widetilde{\Delta}_{\mathcal{R}}$ be the natural extension of Rauzy–Veech induction, where $\widetilde{\Delta}_{\mathcal{R}}$ is the associated space of zippered rectangles. Let ψ_t be the canonical suspension flow associated to $\widetilde{T}$. And g_t the Teichmüller flow on the stratum.

Theorem (Veech)**.** *For every connected component of a normalized strata of Abelian differentials $\mathcal{H}_1$, there exists a connected component in Rauzy diagram $\mathcal{R}$ and a finite-to-one map $\pi_{\mathcal{R}} : \widetilde{\Delta}_{\mathcal{R}} \mapsto \mathcal{H}_1$ such that $\pi_{\mathcal{R}} \circ \psi_t = g_t \circ \pi_{\mathcal{R}}$.*

The Lebesgue measure induces by pull back an invariant measure on $\widetilde{\Delta}_{\mathcal{R}}$ and integrating along fibers of the suspension an invariant measure μ on $\Delta_{\mathcal{R}}$ absolutely continuous with respect to Lebesgue measure.

Corollary 3.12. *The Masur–Veech measure on normalized strata of Abelian differentials is finite and it is the unique measure of maximal entropy for the Teichmüller flow and its entropy is equal to $|\mathcal{A}|$.*

Quadratic differentials The family of linear forms defined by $\delta_{(v,w,\epsilon)}(\lambda) = \lambda(v) - \lambda(w)$ is invariant with respect to the induction. As we saw in Proposition 2.8, the kernel of this linear form intersects the positive cone if and only if the linear involution is balanced. As a consequence we have.

Proposition 3.13. *For linear involutions without flips, the induced map on the subgraph $\mathcal{F}_n$ satisfies hypothesis (Leb^H).*

Proposition 5.2 in [BL09] and results of Section 3 in [Zor08] imply the following result.

Theorem (Boissy–Lanneau, Zorich)**.** *For every connected component of a normalized strata of quadratic differentials $\mathcal{Q}_1$, there exists a connected component in Rauzy diagram $\mathcal{R}$ and a finite-to-one map $\pi_{\mathcal{R}} : \widetilde{\Delta}_{\mathcal{R}} \mapsto \mathcal{H}_1$ such that $\pi_{\mathcal{R}} \circ \psi_t = g_t \circ \pi_{\mathcal{R}}$.*

Corollary 3.14. *The Masur–Veech measure on strata of quadratic differentials is the unique measure of maximal entropy for the Teichmüller flow and its entropy is equal to $|\mathcal{A}| - 1$.*

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