Multidimensional continued fraction algorithms as win-lose inductions

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Abstract

The theory of continued fractions provides a fundamental characterization of the speed of approximation of real numbers by rationals. Generalizing this theory to multiple dimensions has led to the study of numerous piecewise projective linear maps, which encode sequences of rational approximations for vectors in R^d.

In this work, we introduce a unifying framework, the win-lose induction, which interpret these maps as elementary dynamical steps through a labeled graph. We describe a general strategy to associate a win-lose induction to a given multidimensional continued fraction (MCF) allowing us to establish a rigorous bridge between MCF algorithms and Rauzy–Veech induction. By checking a criterion on their corresponding graph, we prove that classical MCF algorithms — including Poincaré, Brun, Selmer, Jacobi-Perron, Skew-product, and Arnoux-Rauzy-Poincaré — admit an invariant ergodic probability measure equivalent to Lebesgue. This measure induces the unique measure of maximal entropy for the associated canonical suspension flow and this flow satisfies an exponential mixing and Central Limit Theorem.

Furthermore, we extend our methods to study fractal sets arising from bounded continued fraction expansions, showing bounds on the Hausdorff dimension of such fractal set strictly smaller than the ambiant space (generalizing results of Berthé–Lee) as well as new explicit bounds on the Hausdorff dimension of Rauzy gaskets.

These results provide a systematic approach to the ergodic analysis of MCFs and their associated fractal structures, strengthening and extending previous works in the field.

Contents

For a real number, the speed of approximation by rational numbers is completely characterized by its continued fraction expansion. From a dynamical perspective that expansion is the coding of an orbit for an elementary dynamical system: the Gauss map $G(x) = \left\{\frac{1}{x}\right\}$. It associates to any irrational number x in $(0,1)$ a sequence of positive integers $a_n := \left\lfloor \frac{1}{G^{n-1}(x)} \right\rfloor$ for $n \geq 1$. The resulting sequence of rational numbers

$$\frac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

converges to x as $n \to \infty$ and provides the best approximations of x, satisfying the property that for all integers $a, b > 0$, if $|bx - a| \leq |q_n x - p_n|$, then $b \geq q_n$. This is a most remarkable phenomenon, central to the study of real numbers.

The theory of multidimensional continued fractions attempts to generalize this phenomenon to the approximation of real vectors by rationals with common denominators. To achieve this, numerous elementary dynamical systems — mostly piecewise projective linear maps — have been introduced and studied. Their ergodic properties are directly linked to the quality of approximation of the sequence of rational vectors they produce.

The first step in generalizing continued fractions is to recognize that the Gauss algorithm arises as the first return map of an even simpler map: the projectivization of the Euclidean algorithm,

$$\Phi : (x,y) \in \mathbb{R}_+^2 \to \begin{cases} (x-y,y) & \text{if } x > y \\ (x,y-x) & \text{if } x < y \end{cases}.$$

As Φ commutes with multiplication by a positive number, its projectivization is often considered to study its dynamics. In projective chart $x \mapsto [x : 1-x]$ the map then becomes the so-called (unsorted) Farey map

$$F : x \in (0,1) \to \begin{cases} \frac{x}{1-x} & \text{if } x < \frac{1}{2} \\ 2 - \frac{1}{x} & \text{if } x > \frac{1}{2} \end{cases}.$$

Many, if not all, Multidimensional Continued Fraction (MCF) algorithms (e.g. every case surveyed in [Sch00]) can be analogously described using similar maps in higher dimensions. These maps usually involve subtracting certain coordinates from others based on their order in size.

Another important perspective on Diophantine approximation is to relate it to the geodesic flow of the modular curve. A geometric generalization of this flow can be made through the moduli spaces of higher genus Riemann surfaces. In this case, the Gauss map generalizes with the Rauzy–Veech induction. This induction is a map on the product of a positive cone with the vertices of a graph. It is also described in an elementary way: it compares two coordinates of the vector in the positive cone and follows one of the

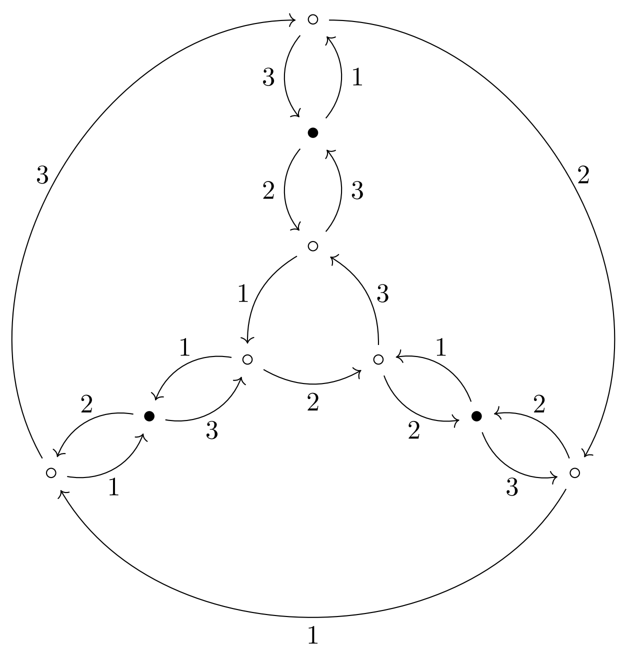


Figure 1: Brun algorithm as a win-lose induction

two outgoing edges of the considered graph vertex according to the order of these two coordinates; the larger is called the winner and the other the loser.

These maps serve as inspiration for the new framework we propose, the *win-lose induction* which generalizes these elementary steps on a graph. The heart of this work is thus to build a bridge between these two transverse perspective, enabling us to bring the rich ergodic techniques that have been developed for moduli spaces of translation surfaces in the past decades to the field of multidimensional continued fractions.

We explain a general strategy to associate a graph defining a win-lose induction to a MCF and compute them for most classical algorithms. As an example, the graph for the Brun algorithm in dimension 3 is depicted in Figure 1.

In [Fou25], a criterion has been developed on the graph which implies strong ergodicity properties on the MCF. The key object to consider are degenerate subgraphs which are build as subgraph where a subset of labels $\mathcal{L}$ is preferred over the other for outgoing edges. In other terms, the degenerate subgraph associated to $\mathcal{L}$ consists in removing outgoing edges not labeled in $\mathcal{L}$ when there is a choice for an edge labeled in $\mathcal{L}$.

The criterion then essentially checks that extremal strongly connected components of degenerate subgraphs do not contain a vertex with more than one outgoing edge labeled in $\mathcal{L}$. In the case of Brun algorithm strongly connected components of degenerate subgraphs of Brun are simple loops on two vertices (see Figure 2) which implies in particular the criterion that we call *non-degenerating property*.

Through various studies, many ergodic theory tools have been introduced to study

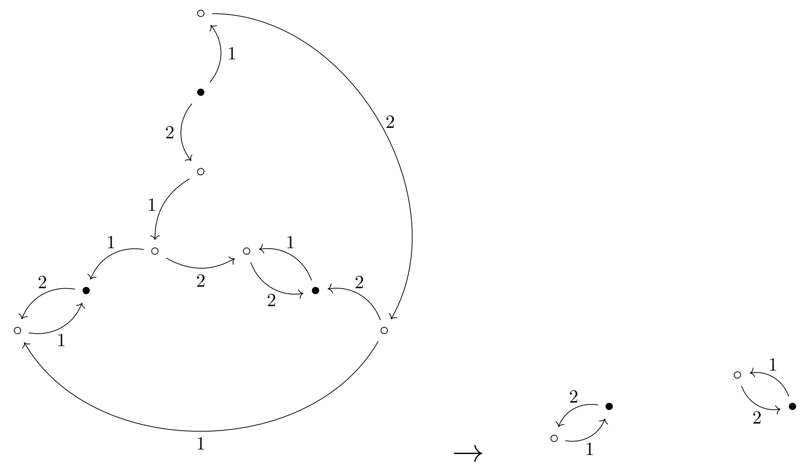


Figure 2: Degenerate subgraph of Brun for $\mathcal{L} = \{1, 2\}$ on the left and its strongly connected components (with multiple vertices) on the right

these approximations. However, these techniques require verifying ad hoc properties for each of these maps and need refined considerations on the shapes of their cylinders (see for instance [Sch00]) which makes them not straightforward to adapt to new examples. The framework developed in this work encompasses all these maps and introduces a criterion which is straightforward to check and that implies strong common ergodic properties for all these examples. We address the so-called Poincaré, Brun, Selmer, Jacobi-Perron, Skew-products, and Arnoux-Rauzy-Poincaré maps, thus covering all the examples listed in Schweiger's reference book [Sch00], strengthening the results of [Sch79], [Sch00], [BAG01].

Theorem. *Poincaré, Brun, Selmer, Jacobi-Perron and Skew-product algorithms in all dimensions as well as Arnoux–Rauzy–Poincaré algorithm admit an invariant ergodic probability measure equivalent to Lebesgue, and this measure induces the unique measure of maximal entropy for the associated canonical suspension flow.*

The generality of this framework also allows for the study of fractal sets associated with these maps, such as the set of points whose continued fractions expansion are bounded by a constant — notably studied by [BL23] — or the Rauzy gasket. We present bounds on the Hausdorff dimension of these objects, with in particular new explicit bounds for the Rauzy gaskets, improving those obtained by Avila-Hubert-Skripchenko in [AHS16].

Theorem. *For Selmer, Jacobi-Perron, and Skew-products, the set of real vectors whose continued fraction expansion is bounded by a constant has a Hausdorff dimension strictly less than the dimension of the ambient space.*

Theorem. *If $\mathcal{G}^d$ denotes the Rauzy gasket in dimension d, we have the bounds*

$$\begin{aligned} \dim_H(\mathcal{G}^2) &< 1.825, \\ \dim_H(\mathcal{G}^3) &< 2.7, \\ \dim_H(\mathcal{G}^4) &< 3.612 \end{aligned}$$

and an asymptotic bound for d going to infinity

$$\dim_H(\mathcal{G}^d) < d - 1 + \frac{\log d}{\log 2 \cdot (d+1)} + o(d^{-1.58}).$$

This also implies new strong ergodic properties for this algorithms. Which where proved for Rauzy–Veech induction in [Buf06] and [AGY06].

Theorem. *In each of these cases, the canonical suspension flow is exponentially mixing and satisfies a Central Limit Theorem.*

We also prove a Bowen-like result to interpret the measure of maximal entropy as the limit of Dirac measures on periodic orbits. The statement of these results will be made precise after introducing our framework in Section 1.

Content of the article In Section 1 we define the win-lose induction associated to a labeled graph as well as a criterion on the graph called non-degenerating property. In a companion work [Fou25], we prove that this criterion implies rich ergodic properties of the associated map. Moreover, fractal sets naturally appear in this setting by studying the parameters that remain confined to a subgraph. We have introduce a similar criterion on these subgraphs to bound the Hausdorff dimension of these sets.

In Section 2, we start by associating a graph to two simpler examples, the so-called Fully-subtractive and Poincaré maps, for which the image of the definition domains for the linear maps is the entire positive cone. These maps are representative examples that do not satisfy the non-degenerating property. In fact, their dynamics are not ergodic and generically get eventually trapped into a subgraph (this is only known in dimension 3 for Poincaré map).

In Section 3, we compute graphs for algorithms whose definition domains are not sent surjectively to the whole simplex by the map, and prove our criterion on them. We show how to compute the graph and its associated subgraph for the fractal set of numbers whose continued fractions are bounded. In the end of the section, we introduce a graph associated with a self-similar fractal constructed with piecewise projective linear maps, the Rauzy gasket.

1 Definitions

1.1 Win-lose induction

Let $G = (V, E)$ denote a finite graph labeled on an alphabet $\mathcal{A}$ by a function $l : E \to \mathcal{A}$. For every vertex $v \in V$, we denote by E_v the subset of edges in E whose start point is v. We assume that the restriction of l to E_v is injective.

Let V^0 be the subset of vertices in V with no outgoing edges. For all v in $V \setminus V^0$ the family of sets indexed by edges $e \in E_v$, and defined by

$$\mathcal{K}^e := \left\{ (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}_+^{\mathcal{A}} \mid \lambda_{l(e)} < \lambda_\alpha \text{ for all } \alpha \in l(E_v) \text{ and } \alpha \neq l(e) \right\},$$

forms a partition of $\mathbb{R}_+^{\mathcal{A}}$ where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$. Additionally, we associate to each edge $e \in E$ a matrix

$$M_e := \mathrm{Id} + \sum_{\substack{\alpha \in l(E_v) \\ \alpha \neq l(e)}} N(\alpha, l(e)).$$

Where $N(a,b)$ is the elementary matrix with coefficient 1 at row a and column b. Such that $\mathcal{K}^e = M_e \cdot \mathbb{R}_+^{\mathcal{A}}$. See Figure 3 for a image of the trace of $\mathcal{K}^a$ on the two dimensional simplex with normalized sum of coordinates and $\mathcal{A} = \{a,b,c\}$.

The *win-lose induction* associated to the graph G is a map

$$\Theta : \left(V \setminus V^0\right) \times \mathbb{R}_+^{\mathcal{A}} \to V \times \mathbb{R}_+^{\mathcal{A}}$$

defined for every edge e from vertices v to v' and all $\lambda \in \mathcal{K}^e$ by $\Theta(v,\lambda) = (v', \Theta_e(\lambda))$, where

$$\Theta_e : \begin{cases} \mathcal{K}^e & \to & \mathbb{R}_+^{\mathcal{A}} \\ \lambda & \mapsto & M_e^{-1}\lambda \end{cases}.$$

These maps are not well defined on the boundaries of subcones $\mathcal{K}^e$. However, we overlook this detail by referring to the entire space as its domain of definition since our primary concern lies in their Lebesgue generic dynamics.

Consider a vertex v with two or more outgoing edges and a parameter $\lambda \in \mathbb{R}_+^{\mathcal{A}}$. In analogy with Rauzy–Veech induction (see [Yoc10] for an introduction), we call **loser** of **losing edge** the edge e such that $\lambda \in \mathcal{K}^e$. Conversely, any other edge e' in E_v is called a **winner**, and we say it wins against e. We sometimes say a label wins or loses when there is no ambiguity to which edge they correspond.

The map Θ can be characterized as follows: it compares the coordinates of all edges emanating from a given vertex v on the vector and subtracts the smallest coordinate from the others it was compared to; effectively subtracting the losing coordinate from the winning ones.

Let us consider the projectivization relation $x \sim \lambda x$ satisfied for all $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}_+^{\mathcal{A}}$. We denote by $\Delta := \mathbb{R}_+^{\mathcal{A}}/\sim$ the simplex of dimension $|\mathcal{A}|-1$ and, for each $v \in V$, by $\{\Delta_e\}_{e\in E_v}$ its induced partition by $\{\mathcal{K}^e\}_{e\in E_v}$. The maps Θ_e can be quotiented by this relation and we denote the induced map by $T_e : \Delta_e \to \Delta$. Similarly, Θ induces a map on spaces $\Delta^+(G) = \left(V \setminus V^0\right) \times \Delta$ and $\Delta(G) := V \times \Delta$ denoted by $T : \Delta^+(G) \to \Delta(G)$.

To study iterates of T, we consider the parameter space $\Delta^\infty(G) := \bigcap_{n\in\mathbb{N}} T^{-n}\Delta(G)$. Notice that if G is strongly connected, then, up to a zero measure subset, $\Delta^+(G) = \Delta(G) = \Delta^\infty(G)$.

Roof function Let us define the *roof function* for (almost) all $x \in \Delta^\infty(G)$ as follows: Let e be the edge from vertex v to v' such that $x \in \{v\} \times M_e\Delta_{v'}$, we set

$$r(x) = -\log\left(\frac{\left|M_e^{-1}x\right|}{|x|}\right).$$

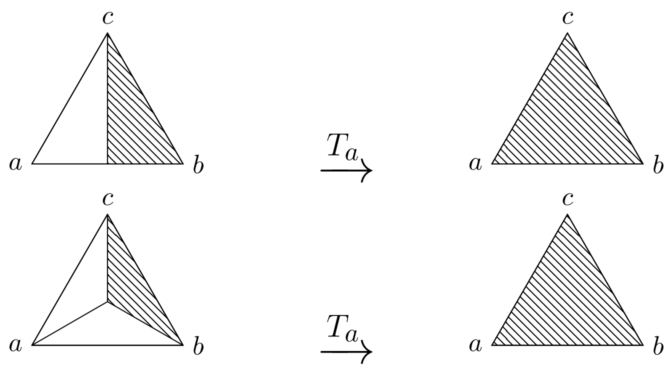


Figure 3: Action of T_a on Δ_a when v has two or three outgoing edges

Canonical suspension flow To define a suspension flow, one need to extend T to a invertible map. Let us consider $\widehat{T} : \widehat{\Delta}(G) \to \widehat{\Delta}(G)$ its natural extension. Notice that in the non-degenerating cases, the space $\widehat{\Delta}(G)$ is isomorphic (up to a zero measure subset) to the set of bi-infinite paths in G.

Define the *suspension space* $\widehat{\Delta}(G)_r := (\widehat{\Delta}(G) \times \mathbb{R})/\sim$, where for all $(x,t) \in \Delta(G) \times \mathbb{R}$ we have the equivalence $(x,t) \sim (Tx, t + r(x))$. The associated *suspension flow* is defined on $\widehat{\Delta}(G)_r$, for all $t \in \mathbb{R}$, by

$$\phi_t : (x,s) \to (x, s+t).$$

Notice that this flow is defined such that the first return map to the section $\widehat{\Delta}(G) \times \{0\}$ is T and its return time is r.

Denote by $\mathcal{M}_{T,r}$ the set of T-invariant Borel probability measures with $\mu(r) := \int_{\Delta(G)} r d\mu < +\infty$. Every ϕ_t-invariant Borel probability measure $\widetilde{\mu}$ on $\Delta(G)_r$ can be decomposed as a product of (the extension to $\widehat{\Delta}(G)$ of) a measure $\mu \in \mathcal{M}_{T,r}$ and the Lebesgue measure on fibers. Namely,

$$\widetilde{\mu}_r = (\mu(r))^{-1} (\mu \times \mathrm{Leb})_{|\Delta(G)_r}.$$

The Kolmogorov–Sinai entropy of the flow for this measure is written $h(\phi, \widetilde{\mu})$ and satisfies Abramov's formula

$$h(\phi, \widetilde{\mu}) = \frac{h(T,\mu)}{\mu(r)}$$

where $h(T,\mu)$ is the Kolmogorov–Sinai entropy for T. In this setting the topological entropy can be defined as

$$h_{\mathrm{top}}(\phi) = \sup_{\mu \in \mathcal{M}_{T,r}} h(\phi, \widetilde{\mu}_r).$$

The induced measure $\widetilde{\mu}_r$ for $\mu \in \mathcal{M}_{T,r}$ at which this supremum is achieved (and by extension μ itself) is referred to as a *measure of maximal entropy.*

Non-degenerating property Assume on a non-trivial subset of labels $\mathcal{L} \subset \mathcal{A}$ the parameter in $\mathbb{R}_+^{\mathcal{A}}$ have coordinates in $\mathcal{L}$ infinitely smaller than others. When we are at a vertex that has one outgoing edge labeled in $\mathcal{L}$, any edge not labeled in $\mathcal{L}$ must win. Hence, the map T will remain in a subgraph in which we remove such edges not labeled in $\mathcal{L}$.

This motivates the introduction of the *degenerate subgraph* $G_{\mathcal{L}}$ having the same set of vertices V as G but for which we remove edges along which a letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$. In other words, the set of outgoing edges for a vertex $v \in V$ in $G_{\mathcal{L}}$ is defined as follows.

- If $l(E_v) \cap \mathcal{L} \neq \emptyset$
$$E_v^{\mathcal{L}} = \{e \in E_v \mid l(e) \in \mathcal{L}\}.$$
- Otherwise
$$E_v^{\mathcal{L}} = E_v.$$

Definition 1.1. *We say that the base graph of a win-lose induction is* non-degenerating *if it is strongly connected and, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and all vertices v in a strongly connected component $\mathscr{C}$ of $G_{\mathcal{L}}$, one of the following properties holds:*

1. *There is a path from v in G labeled in $\mathcal{L}$ leaving $\mathscr{C}$.*
2. *$|l(E_v) \cap \mathcal{L}| \leq 1$.*

In plain words: from any vertex, no letter in $\mathcal{L}$ can win against another letter in $\mathcal{L}$ in any strongly connected component of $G_{\mathcal{L}}$ except if there is a path labeled in $\mathcal{L}$ leaving the component.

It is easy to check that this property is satisfied by Rauzy diagrams (see [Yoc10] for an introduction). Our goal in this work is to prove it for classical multidimensional continued fractions algorithms. We then can apply one of the main theorem in [Fou25].

Theorem A. *Every non-degenerating win-lose induction has a unique invariant measure equivalent to Lebesgue measure for its projectivized map and it induces the unique invariant probability measure of maximal entropy for the flow on its canonical suspension.*

The entropy of this canonical suspension flow is equal to $|\mathcal{A}| - 1$ where $|\mathcal{A}|$ is the number of coordinates in the projective vector of the map.

Remark. *In this article, we will only refer to ergodicity for the projectivized map. We will not address the question of ergodicity for the homogeneous map as studied in [CN13]. Hence when talking about the ergodic measure of a MCF we will always imply an ergodic measure for its projectivized map.*

Fractal set associated to a subgraph Let $F = \left(V^F, E^F\right)$ be a subgraph of a win-lose induction base graph G. Define

$$H(F,G) := \bigcup_{v \in V^F} \{v\} \times \left(\bigcup_{e \in E_v^F} M_e \Delta \right)$$

and

$$\Delta^\infty(F,G) := \bigcap_{n\in\mathbb{N}} T^{-n}H(F,G)$$

the set of parameters in the simplices for which the win-lose induction remains in the subgraph F.

In the graphs associated to multidimensional continued fraction algorithms that we compute in this articles, all vertices will have multiple outgoing edges. Nevertheless, for subgraphs F associated to fractal sets, some vertices with a unique outgoing edge will appear. From a dynamical perspective, such vertices can be skipped in the orbit of a win-lose induction until we meet a *branching vertex*, *i.e.* a vertex with several outgoing edges. A path in F for which the start and end vertices, and only them two, are branching vertices is called a *branch path*. We label such a path by the label of its first edge.

And we have a generalization of the non-degenerating property to this setting.

Definition 1.2. *Consider a graph G defining a win-lose induction and F a strongly connected subgraph such that the removed edges from G are going out of a vertex with only one outgoing edge in F. We say that F is an admissible subgraph if for any $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ and every branch path γ contained in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$, starting at a vertex v, one of these properties is true.*

1. *There exists a path within F starting at a vertex of γ and leaving $\mathscr{C}$ such that each edge based at a branch vertex is labeled in $\mathcal{L}$.*

2. *All edges in γ are labeled by letters not in $\mathcal{L}$.*

3. *The branch path γ is labeled by $\alpha \in \mathcal{L}$ and $\mathcal{L} \cap l(E_v^F) = \{\alpha\}$. Moreover, for every edge of F in γ labeled in $\mathcal{L}$, the removed edges going out of its starting vertices are labeled in $l(E_v^F) \setminus \{\alpha\}$.*

In the case of Rauzy gasket, it will be necessary to factor a piece of path *before* the branching vertex and to define a different factorization for each degenerate subgraph, depending on the set $\mathcal{L}$. We thus introduce the last and more general non-degenerating property for this particular application.

Assume that, for all $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, we are given $\widetilde{V}_{\mathcal{L}}$, a subset of vertices of $G_{\mathcal{L}}$, such that each path γ in $G_{\mathcal{L}}$ between two of these vertices can be uniquely decomposed as a finite concatenation $\gamma_1 \dots \gamma_n$ where each γ_i is a path with only its start and end vertices in $\widetilde{V}_{\mathcal{L}}$. Moreover, the paths γ_i visit at most one (excluding its ending vertex) branching vertex which has an outgoing edge labeled in $\mathcal{L}$.

Such paths γ_i appearing in the decomposition are called $\mathcal{L}$-factor paths. We denote by $\widetilde{E}^{\mathcal{L}}$ their set and $\widetilde{E}_v^{\mathcal{L}}$ the subset of paths starting at vertex v.

Notice that $\widetilde{E}^{\mathcal{L}}$ is the set of paths in G between vertices in $\widetilde{V}_{\mathcal{L}}$ along which no letter in $\mathcal{L}$ wins against a letter not in $\mathcal{L}$.

Remark. *The branch paths are a particular case of factor path where $\widetilde{V}_{\mathcal{L}}$ is given by the set of branching vertices.*

Definition 1.3. *Consider a graph G defining a win-lose induction and F a strongly connected subgraph such that the removed edges from G are going out of a vertex with only one outgoing edge in F. We say that F has an* admissible factorization *if for any $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$ there exists a subset of vertices $\widetilde{V}_{\mathcal{L}} \subset V$ as above such that for every $\mathcal{L}$-factor path $\gamma \in (\widetilde{E}^F)^{\mathcal{L}}$ in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$ one of the properties 1 and 2 in Definition 1.2 is true or*

3. *There is an edge in γ labeled $\alpha \in \mathcal{L}$ and starting at a branching vertex $v \in \widehat{V}^F$ such that $\mathcal{L} \cap l(E_v^F) = \{\alpha\}$. Moreover, for every edges in γ labeled in $\mathcal{L}$, the removed edges going out of its starting vertices are labeled in $l(E_v^F) \setminus \{\alpha\}$.*

The following theorem was proved in Section 2.2.4 of [Fou25].

Theorem B. *If F is an admissible subgraph — or a subgraph with an admissible factorization — of a graph G defining a win-lose induction labeled on $\mathcal{A}$ then the Hausdorff dimension of the associated fractal set is strictly smaller than the dimension of its ambiant space*

$$\dim_H \Delta^\infty(F, G) < |\mathcal{A}| - 1.$$

More precisely,

$$\dim_H \Delta^\infty(F, G) \leq |\mathcal{A}| - 1 - \frac{|\mathcal{A}| - \kappa_0}{|\mathcal{A}|}$$

where $\kappa_0 < |\mathcal{A}|$ is the entropy of the canonical suspension flow of the induction.

Other strong ergodic results are proved in [Fou25] for the first return map of a win-lose algorithms to a compactly supported subsimplex and its associated suspension flow and flow. Namely, an exponential mixing property, a Central Limit Theorem and a construction of the measure of maximal entropy as a limit of sum of Dirac measures on periodic orbits. We refer the reader to sections 3.3.1 and 3.5 in [Fou25] for precise statements.

1.2 Induced n-path graph

A multiplicative version of continued fraction algorithms is often considered, where we iterate the additive algorithm as long as the definition domain (or the vertex in our description) does not change. The quintessential example for this is Gauss algorithm which is a multiplicative version of the Euclide algorithm applied until the order of the two coordinates changes.

In order to describe such multiplicative algorithms as the first return map of a win-lose induction, one needs to define a cover of the graph of the additive algorithm which keeps track of the previous vertex in a path. That is a motivation for the following definition.

Consider a win-lose induction on a base graph $G = (V, E)$ and a subset of vertices $V_\circ$, corresponding to the domains of definition for the algorithm. We sometime call the vertices in $V_\circ$ *state vertices*. Assume any loop in the graph must contain a vertex in $V \setminus V_\circ$ thus every path eventually goes through a vertex in $V_\circ$ after a bounded number of steps.

For $n \in \mathbb{N}^*$, denote by $G^n = (V^n, E^n)$ the induced n-path graph of G, which keeps track of the n last visited vertices in $V_\circ$, defined as follows.

- V^n is the set of finite paths in G that start at a vertex in $V_\circ$ and visit exactly $n+1$ vertices in $V_\circ$. Any path in V^n can be decomposed as $\gamma_1 \dots \gamma_n \gamma_f$, where:
  - For all $1 \leq i \leq n$, γ_i is a non-empty path in which only the starting and ending vertices are in $V_\circ$.
  - γ_f is a path, possibly empty, in which only the starting vertex is in $V_\circ$.

  The no loop condition for the induced graph on $V \setminus V_\circ$ implies that V^n is finite.

- There is an edge from $\gamma = \gamma_1 \dots \gamma_n \gamma_f$ to $\gamma' = \gamma'_1 \dots \gamma'_n \gamma'_f$ if and only if there exists an edge e in E such that either:
  - $\gamma e = \gamma'$, or
  - $\gamma_2 \dots \gamma_n \gamma_f e = \gamma'_1 \dots \gamma'_n$ and γ'_f is empty.

  In the second case, we have $\gamma_2 = \gamma'_1, \dots, \gamma_n = \gamma'_{n-1}$, and $\gamma_f e = \gamma'_f$.

- The labeling $l : E^n \to \mathcal{A}$ assigns to each edge between ν and ν' the label of the last edge of the path ν'. Thus, G^n is also labeled by $\mathcal{A}$.

Remark 1.4. *When studying continued fractions, a natural fractal set arises from the parameters whose expansion is bounded by a given integer n. This corresponds to paths in G^n that do not loop on a vertex of the form $v = \gamma_1 \gamma_1 \dots \gamma_1$. The fractal set is defined by the subgraph F^n, where we remove, for every such vertex (where γ_1 can be decomposed as $\gamma_1 = \gamma_s e$, with e being the last edge of the path), the edge between $\gamma_1 \dots \gamma_1 \gamma_s$ and $\gamma_1 \dots \gamma_1$.*

Let $\pi_n : G^n \to G$ be a graph homomorphism given by:

- A map $(\pi_n)_V : V^n \to V$ which associates to a path in V^n its ending vertex,
- A map $(\pi_n)_E : E^n \to E$ which associates to an edge in G^n from ν to ν' the last edge of the path ν'.

For simplicity in the notation we denote both maps by π_n where there is no ambiguity. Both maps are surjective and for every vertex in $\nu \in V^n$ there is a bijection, with matching labels, between outgoing (resp. ingoing) edges of ν in G^n and outgoing (resp. ingoing) edges of $(\pi_n)(\nu)$. We say that G^n is a covering graph of G. It implies in particular the following key property.

Property. *Any path γ in G starting at $v \in V$ can be lifted to a path $\widetilde{\gamma}$ in G^n starting from any point in $\pi_n^{-1}(v)$ and such that $\pi_n(\widetilde{\gamma}) = \gamma$.*

For a vertex $v \in V$, denote by $\mathscr{C}_v$ the strongly connected component of v in G formed by vertices v in V such that there is a path from v to v' and from v' to v as well as the edges between them. We define similarly the strongly connected components of vertices in V^n. For any ν in V^n, $\pi_n(\mathscr{C}_\nu) \subset \mathscr{C}_{\pi_n(\nu)}$ and the labels of outgoing edges of ν in $\mathscr{C}_\nu$ are the same as for $\pi_n(\nu)$ in $\mathscr{C}_{\pi_n(\nu)}$.

Lemma 1.5. *If G is non-degenerating then so is G^n for all $n \in \mathbb{N}^*$.*

Proof. Let us start by showing strong connectivity of G^n. Let ν and ν' be two vertices of G^n. By construction, ν' corresponds to a path γ in G starting at a vertex denoted by $v \in V$. By strong connectivity of G, there exists a path γ_s from $\pi_n(\nu)$ to v. Then by definition, the path $\gamma_s \cdot \gamma$ lifts to a path in G^n from ν to ν'.

Let $\mathcal{L}$ be a non-trivial subset of the alphabet $\mathcal{A}$ and $G^n_{\mathcal{L}}$ the corresponding degenerating subgraph of G^n. Consider a strongly connected component $\mathscr{C}$ of this subgraph. A vertex ν in $\mathscr{C}$ corresponds to a path from a vertex v to a vertex $v' = \pi_n(\nu)$ in G. If it satisfies $|l(E_{v'}) \cap \mathcal{L}| \leq 1$ so does ν since their outgoing edges have the same labels. If there exists a path in G from v' labeled in $\mathcal{L}$ and leaving $\mathscr{C}_{v'}$, this path can be lifted to a path in G^n starting at ν and as $\pi_n(\mathscr{C}) \subset \mathscr{C}_{v'}$ is must leave $\mathscr{C}$. □

2 Two full-image examples

Let $\Phi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a map associated with a MCF algorithm, where Φ is linear on each tile of a partition of $\mathbb{R}^n_+$ into subcones $D_1, \ldots, D_k$. We begin by computing the associated win-lose induction for two classical examples, both of which satisfy the simplifying condition that for all $i \in \{1, \ldots, k\}$, $\Phi(D_i) = \mathbb{R}^n_+$.

2.1 Fully subtractive algorithms

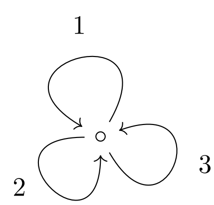
The fully subtractive algorithm in dimension 3 can be described by the map, defined at almost every point by

$$\Phi : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \to (x'_1, x'_2, x'_3) \in \mathbb{R}^3_+,$$

where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i > x_j > x_k$,

$$x'_i = x_i - x_k, \quad x'_j = x_j - x_k, \quad x'_k = x_k.$$

This map corresponds to a step for the win-lose induction in the graph with one vertex and three edges of distinct labels, represented below.



This construction generalizes to fully subtractive algorithms in dimensions $n > 3$ by considering a single vertex with n loops, each labeled by a different letter.

None of these algorithms satisfy the non-degenerating property. They correspond to the stable subgraphs phenomenon described in Section 2.2.1 of [Fou25], where almost every orbit eventually becomes trapped in a degenerate subgraph, preventing the projectivized map from being ergodic. One coordinate remains significantly larger than the other two, which decrease rapidly under the application of a continued fraction algorithm. A similar behaviour appears for the 3-dimensional Poincaré algorithm discussed in the next subsection,

2.2 Poincaré algorithms

Poincaré algorithm has been introduced by Poincaré as a generalization of the continued fraction algorithm and was later studied and generalized in [Nog95]. It can be described by the map

$$\Phi : (x_1, x_2, x_3) \in \mathbb{R}_+^3 \to (x_1', x_2', x_3') \in \mathbb{R}_+^3,$$

where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i > x_j > x_k$,

$$x_i' = x_i - x_j, \quad x_j' = x_j - x_k, \quad x_k' = x_k.$$

This map corresponds to the first return map of the win-lose induction represented on Figure 4 to the white node (where all white nodes are identified). The first step is determining which coordinate is the smallest of the three and subtracting it to the other two. The second step is comparing the two initially largest coordinates and subtracting the smallest to the largest. This is precisely describing Poincaré algorithm.

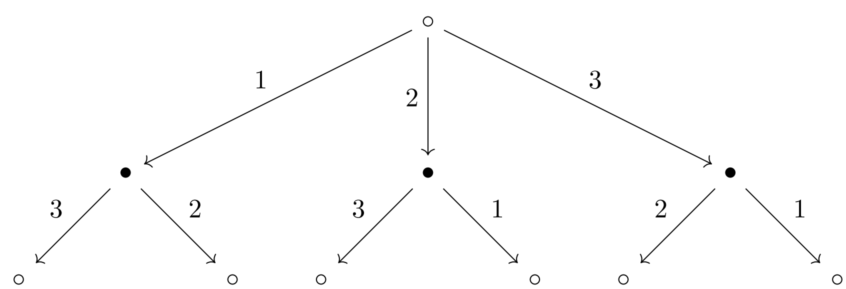


Figure 4: Poincaré algorithm as a win-lose induction

The win-lose induction induced on the subgraph $G_{\{1,2\}}$, where G is the graph depicted in Figure 4, is a perturbed Euclidean algorithm as studied in Section 2.2.1 of [Fou25]. Similar to fully subtractive algorithms, almost every orbit eventually becomes trapped in such a subgraph. This phenomenon which was observed by Nogueira [Nog95] 30 years ago with some surprise from the community.

For the higher-dimensional Poincaré algorithm, this tree graph construction generalizes to dimension n by starting with a vertex having n outgoing edges labeled by distinct letters. We proceed iteratively. For each edge pointing to a leaf of the tree, consider the list of every other edges going out of its start vertex and add them as the outgoing vertices of its end vertex. When the leaves has only edge point to them, we stop and identify them with the root.

The cases of dimensions $n \geq 4$ are the only classical examples to our knowledge for which the non-degenerating property is not satisfied but does not have obvious stable subgraphs. Weather the projectivized maps for these algorithms are ergodic or not is still open (ergodicity is conjectured by Nogueira for even n).

3 Other examples

Let now $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a map associated to a MCF algorithm which does not necessarily have full image. Assume there exists a partition of $\mathbb{R}_+^n$ by subcones $D_1, \ldots, D_k$, called domain sets, such that for each $i \in \{1, \ldots, k\}$, Φ restricted to D_i is a linear map. We describe a process to reduce to the previous case when for each $i \in \{1, \ldots, k\}$, $\Phi(D_i)$ is the union of domain sets.

Assume that, for all $i \in \{1, \ldots, k\}$, there exists a matrix $\Pi_i \in \mathrm{GL}(d+1, \mathbb{R})$ such that $\Pi_i \mathbb{R}_+^n = D_i$. These matrices induce a bijection between $\mathbb{R}_+^n \times \{1, \ldots, k\}$ and $\bigsqcup_{i=1}^k D_i$ defined almost everywhere. Let ι be the inclusion map from $\bigsqcup_{i=1}^k D_i$ to $\mathbb{R}_+^n$ and π be defined by the following commutative diagram. The map π can be projectivized by identifying positive lines in the cone, it induces a bijection $p : \Delta \times \{1, \ldots, k\} \to \Delta$.

$$\begin{array}{ccc} \mathbb{R}_+^n \times \{1, \ldots, k\} & \xrightarrow{\simeq} & \bigsqcup_{i=1}^{k} D_i \\ & \searrow^{\pi} & \big\downarrow \iota \\ & & \mathbb{R}_+^n \end{array}$$

We define an induced map $\widetilde{\Phi}$ on $\mathbb{R}_+^n \times \{1, \ldots, k\}$ by

$$\widetilde{\Phi}(x, i) = \left(\Pi_j^{-1} \cdot \Phi(\Pi_i \cdot x), j\right)$$

when $\Pi_i \cdot x \in D_j$. This is well defined almost everywhere since subcones D_j partition $\mathbb{R}_+^n$.

By construction, we have $\pi \circ \widetilde{\Phi} = \Phi \circ \pi$ and the image sets of $\widetilde{\Phi}$ are full cones $\mathbb{R}_+^n \times \{j\}$. Our goal is then to construct as in the previous section a win-lose induction graph for which a first return map to a given set of vertices of the win-lose induction map is conjugated to $\widetilde{\Phi}$. The semi-conjugated map Φ will then share the same ergodic properties.

Let us consider a graph which vertices are labeled by the set $\{1, \ldots, k\}$ and such that there is an edge from vertex k to vertex l if and only if D_l is a subset of $\Phi(D_k)$. This graph is called the *combinatoric graph* of domain sets.

On can think of the map $\widetilde{\Phi}$ as acting on this graph similarly to a win-lose induction. At a given vertex, it splits each simplex it into subsimplices, corresponding to outgoing edges, and maps each part into the whole simplex corresponding to the end vertex of that edge via a projective linear transformation.

For each vertex, we then look for description of the splitting as a win-lose induction on a tree. The root simplex is split along its partition into domains D_j, and the edges leading to the leaves are connected to the corresponding image simplices as determined by the combinatoric graph.

One might argue that the splitting of simplices alone is insufficient information, as the map acts as a projective linear transformation on each domain. However, in the cases we are considering, the inverse branches of $\widetilde{\Phi}$ are projective linear maps defined by non-negative integer matrices in $\mathrm{SL}(d+1, \mathbb{Z})$.

The following proposition demonstrates that, in these cases, the win-lose induction we have defined is conjugate, up to a permutation of the extremal points of the simplices, to the map $\widetilde{\Phi}$.

Proposition 3.1. *If two non-negative matrices in* $\mathrm{SL}(d+1, \mathbb{Z})$ *have the same projective action on the extremal points of* Δ*, then the matrices are equal.*

Proof. Let $v_1, \ldots, v_{d+1}$ be the vectors defining the extremal points of Δ. Suppose the first matrix maps these vectors to $w_1, \ldots, w_{d+1}$.

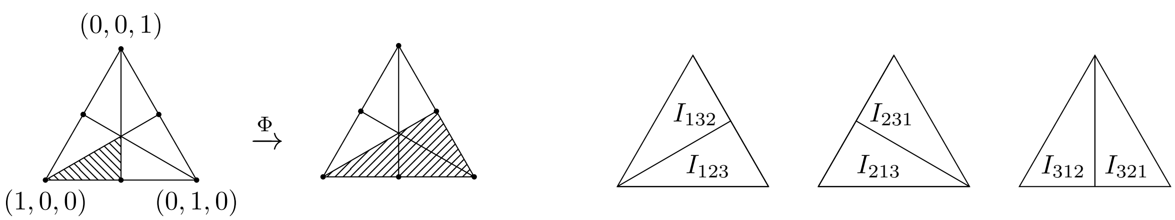
By assumption, the second matrix maps the same vectors to $\alpha_1 w_1, \ldots, \alpha_{d+1} w_{d+1}$, where $\alpha_1, \ldots, \alpha_{d+1}$ are non-negative scalars. Since both matrices are in $\mathrm{SL}(d+1, \mathbb{Z})$ and have determinant 1, we also have the condition $\prod_{k=1}^{d+1} \alpha_k = 1$. Which implies that $\alpha_1 = \cdots = \alpha_{d+1} = 1$. □

3.1 Brun algorithms

Dimension 3 The Brun algorithm, introduced by Brun in 1957, is described in dimension 3 by the map $\Phi : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \to (x'_1, x'_2, x'_3) \in \mathbb{R}^3_+$ where, if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i > x_j > x_k$,

$$x'_i = x_i - x_j, \quad x'_j = x_j, \quad x'_k = x_k.$$

The domains of definition for this map are given by the order of coordinates by size: we label them by ijk when $x_i > x_j > x_k$.



(a) Projective image of D_{123} (b) All image sets, where $I_{ijk} = \Phi(D_{ijk})$

Matrices for domains sets are defined as follows.

$$\Pi_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Pi_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Pi_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Pi_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \Pi_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Pi_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The associated combinatoric graph for this MCF, as introduced at the head of this section, is represented in Figure 6. The states in dashed circles are identified with the states of same label.

We associate to $\Pi_{123}(D_{123})$ a tree inducing the same splitting. By symmetry we define a graph in Figure 8, where the dashed arrow on left and right are identified.

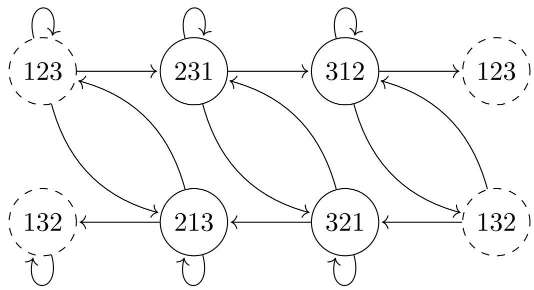
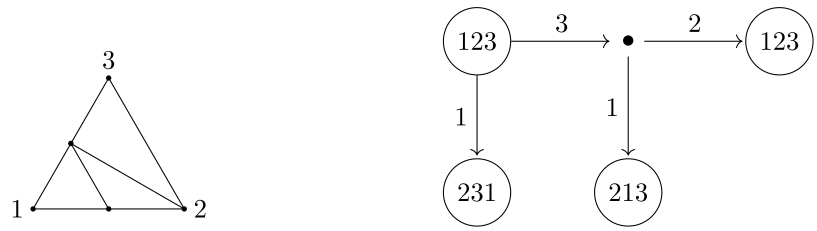


Figure 6: Combinatoric graph of Brun algorithm domain sets



(a) Partition of $\Pi_{123}^{-1}(123)$ (b) Corresponding win-lose tree

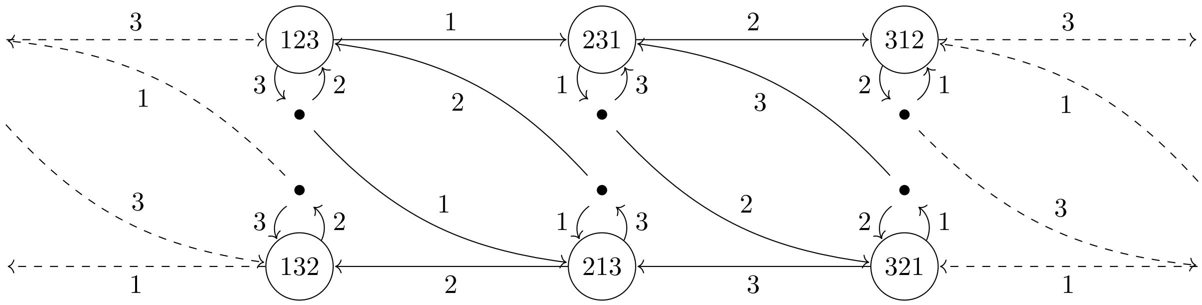


Figure 8: Brun algorithm as a win-lose induction

Proposition 3.2. *Let $\Theta_\circ$ be the first return map of the win-lose induction on the graph in Figure 8 to the white circle vertices, then we have $\pi \circ \Theta_\circ = \Phi \circ \pi$.*

Remark 3.3. *By symmetry, we can define the same map $\Theta_\circ$ as a win-lose induction on a graph composed of only 9 vertices. Indeed each top black vertex has its vertex labeled and pointing to a vertex exactly as for the black vertex at its bottom right. Thus we can identify these pairs of black vertices to obtain a smaller graph (represented in introduction of [Fou25]) defining the same accelerated algorithm.*

We now check the non-degenerating property. In $G_{\{1,2\}}$ the only non-trivial strongly connected components are two loops around 312 and 321 which are clearly non-degenerating. The same is true for any two letters and implies the following proposition.

Proposition 3.4. *The graph associated to Brun algorithm in dimension 3 is non-degenerating.*

In particular by Theorem A, we obtain an alternative proof of the following result as well as the fact that it induces the unique measure of maximal entropy for the canonical suspension flow.

Theorem ([Sch79], [Sch91])**.** *The Brun algorithm in dimension* 3 *admits an invariant ergodic measure equivalent to Lebesgue measure.*

Higher dimensions This construction can be generalized to all dimensions.

For any $n \geq 2$, the Brun algorithm is defined by the map

$$\Phi : (x_1, \dots, x_n) \in \mathbb{R}_+^n \to (x'_1, \dots, x'_n) \in \mathbb{R}_+^n,$$

where for $\sigma \in \mathfrak{S}_n$ such that $x_{\sigma_1} > \cdots > x_{\sigma_n}$ we have

$$\begin{aligned} x'_{\sigma_1} &= x_{\sigma_1} - x_{\sigma_2} \\ x'_{\sigma_i} &= x_{\sigma_i} \text{ for all } i \geq 2. \end{aligned}$$

The domains of definition are here labeled by permutations in $\mathfrak{S}_n$ and will be correspondingly denoted by D_σ. They are sent projectively on Δ by the following identification

$$y_{\sigma_n} = x_{\sigma_n}, \; y_{\sigma_{n-1}} = x_{\sigma_{n-1}} - x_{\sigma_n}, \; \dots \; , \; y_{\sigma_1} = x_{\sigma_1} - x_{\sigma_2}. \tag{1}$$

The corresponding matrix defines Π_σ^{-1}.

For any permutation $\sigma \in \mathfrak{S}_n$, the image set $\Phi(D_\sigma)$ is defined by the inequalities $x'_{\sigma_2} > \cdots > x'_{\sigma_n}$. In other words, the order of $x_{\sigma_2}, \dots, x_{\sigma_n}$ is preserved, while x'_{σ_1} can occupy any position. Thus $\Phi(D_\sigma) = \bigcup_{k=1}^n \overline{D_{\sigma\cdot(1\dots k)}}$.

In the corresponding combinatoric graph, there are edges from domain vertices σ to all $\sigma \cdot (1 \dots k)$ for $1 \leq k \leq n$.

The splitting of $\Pi_\sigma^{-1} D_\sigma$ can then be described by the following process, corresponding to the tree in Figure 9.

Start by checking whether $y_{\sigma_1} = x_{\sigma_1} - x_{\sigma_2}$ is smaller than $y_{\sigma_n} = x_{\sigma_n}$.

– If it is, Φ maps this part of the domain to $D_{\sigma\cdot(1\dots n)}$.

– If not, we send the other half of the simplex to the whole simplex by considering $y'_{\sigma_1} = y_{\sigma_1} - y_{\sigma_n}$. Now check whether $y'_{\sigma_1} = x_{\sigma_1} - x_{\sigma_2} - x_{\sigma_n}$ is smaller than $y_{\sigma_{n-1}} = x_{\sigma_{n-1}} - x_{\sigma_n}$.

  – If it is, this part is mapped to $D_{\sigma\cdot(1\dots(n-1))}$.

  – If not, consider $y''_{\sigma_1} = y'_{\sigma_1} - y'_{\sigma_{n-1}} = y_{\sigma_1} - y_{\sigma_{n-1}}$ and continue ...

Proposition 3.5. *Consider the win-lose induction defined by the combinatorics graph and the win-lose trees described in Figure 9. Let $\Theta_\circ$ be its first return map to vertices corresponding to image sets. Then we have $\pi \circ \Theta_\circ = \Phi \circ \pi$.*

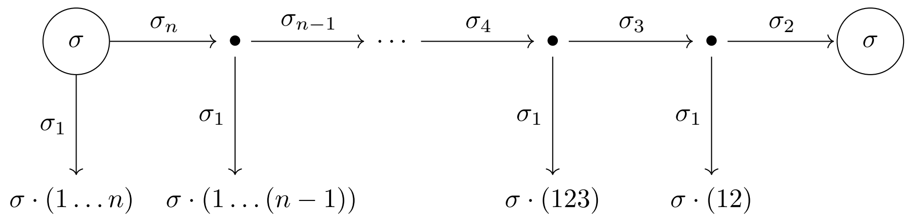


Figure 9: Win-lose tree between domain sets for Brun algorithm

Proposition 3.6. *The graph associated with the Brun algorithm in all dimensions is non-degenerating.*

Proof. Let G denote the corresponding graph which is clearly strongly connected. Notice that when starting from a vertex corresponding to permutation $\sigma \in \mathfrak{S}_n$, the accelerated win-lose induction $\Theta_\circ$ moves to another domain set associated with a permutation σ'. This new permutation σ' is obtained from σ by inserting σ_1 in front of the last winning label.

Let us consider $\mathcal{L}$ a nontrivial subset of $\mathcal{A} = \{1, \dots, n\}$. In the degenerate subgraph $G_\mathcal{L}$, the accelerated induction moves from one permutation to another, where the following quantity does not decrease

$$M(\sigma, \mathcal{L}) := \max\{0 \leq i \leq n \mid \sigma([n-i+1, n]) \subset \mathcal{L}\}.$$

For a given permutation σ, consider the subgraph of the tree in Figure 9 contained $G_\mathcal{L}$.

- If $\sigma_1 \notin \mathcal{L}$, there are no vertices with more than one edge labeled in $\mathcal{L}$.
- If $\sigma_1 \in \mathcal{L}$, the subgraph contains a path composed of edges labeled by $\sigma_n, \sigma_{n-1}, \dots, \sigma_{n-l+1}$ in $\mathcal{L}$, where $l = M(\sigma, \mathcal{L})$.
  - If a vertex is in this path, consider the path labeled in $\mathcal{L}$ going to the end vertex of σ_{n-l+1} and add vertex σ_1. It goes to the vertex labeled by $\sigma \cdot (1 \dots (n-l))$. By definition, $\sigma_{n-l} \notin \mathcal{L}$, so $M(\sigma \cdot (1 \dots (n-l)), \mathcal{L}) > l$. Therefore, this path is labeled in $\mathcal{L}$ and exits the strongly connected component.
  - If a vertex is not contained in this path and is not a vertex corresponding to a permutation then its strongly connected component must be trivial.

□

This implies an alternative proof of another of Schweiger's theorems.

Theorem ([Sch00])**.** *The (sorted) Brun algorithm in all dimensions admits an invariant ergodic measure equivalent to Lebesgue measure.*

A sorted algorithm corresponds to the map Φ, restricted to vectors with ordered coordinates, followed by a permutation that reorders the coordinates after the map is applied. This theorem is proved for a sorted version of Brun algorithm and is implied in our alternative proof by a stronger result on the unsorted algorithm.

Theorem 3.7. *Brun algorithms in all dimensions admit an invariant ergodic measure equivalent to Lebesgue measure which induces the unique measure of maximal entropy of its canonical suspension flow.*

Multiplicative version The Brun algorithm is often studied in the literature in its *multiplicative* version, where one iterates the previous map until the ordering of coordinates changes.

For any $n \geq 2$, the Brun multiplicative algorithm is defined by the map

$$S(x_1, \ldots, x_n) \in \mathbb{R}_+^n \to (x_1', \ldots, x_n') \in \mathbb{R}_+^n$$

where, for $\sigma \in \mathfrak{S}_n$ such that $x_{\sigma_1} > \cdots > x_{\sigma_n}$, we have

$$x_{\sigma_1}' = x_{\sigma_1} - N \cdot x_{\sigma_2} \quad \text{with } N = \left\lfloor \frac{\sigma_1}{\sigma_2} \right\rfloor,$$
$$x_{\sigma_i}' = x_{\sigma_i} \text{ for all } i \geq 2.$$

The first return map induced by the win-lose induction on the 1-path cover of the graph G^1 to the vertices not corresponding to loops in the combinatoric graph is conjugated to S with the same projection maps π.

3.2 Jacobi-Perron and Skew-products

Classically, the Jacobi-Perron algorithm is defined for $n \geq 3$ as the map on $[0,1]^{n-1}$,

$$(x_1, \ldots, x_{n-1}) \mapsto \left(\left\{ \frac{x_2}{x_1} \right\}, \ldots, \left\{ \frac{x_{n-1}}{x_1} \right\}, \left\{ \frac{1}{x_1} \right\} \right).$$

Whereas the skew-product algorithm (often called Ostrowski algorithm for $n = 3$) is defined for $n \geq 3$ as the map on $[0,1]^{n-1}$,

$$(x_1, \ldots, x_{n-1}) \mapsto \left(\left\{ \frac{1}{x_1} \right\}, \left\{ \frac{x_2}{x_1} \right\}, \ldots, \left\{ \frac{x_{n-1}}{x_1} \right\} \right).$$

If we identify the projectivized space $\mathbb{R}^n$ with $\mathbb{R}^{n-1}$ by normalizing the last coordinate, they correspond respectively to the following linear projective maps, defined on the subcone where $x_n \geq x_i$ for all $1 \leq i \leq n$,

$$F\left([x_1 : \cdots : x_{n-1} : 1]\right) = \left[x_2 - \left\lfloor \frac{x_2}{x_1} \right\rfloor x_1 : \ \cdots : \ x_{n-1} - \left\lfloor \frac{x_{n-1}}{x_1} \right\rfloor x_1 : \ 1 - \left\lfloor \frac{1}{x_1} \right\rfloor x_1 : \ x_1 \right]$$

and

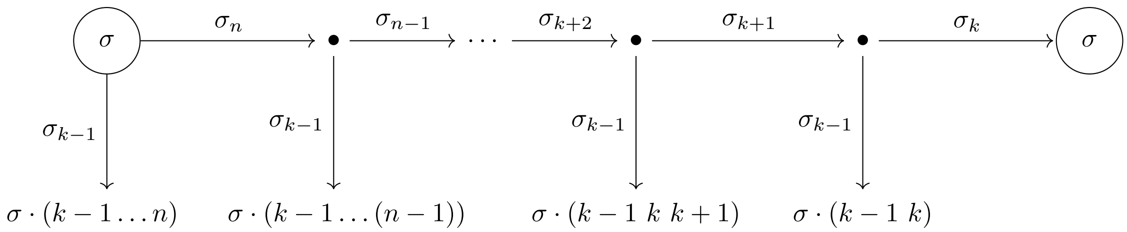
$$F\left([x_1 : \cdots : x_{n-1} : 1]\right) = \left[1 - \left\lfloor \frac{1}{x_1} \right\rfloor x_1 : x_2 - \left\lfloor \frac{x_2}{x_1} \right\rfloor x_1 : \ \cdots : \ x_{n-1} - \left\lfloor \frac{x_{n-1}}{x_1} \right\rfloor x_1 : x_1 \right].$$

They can be interpreted as the projectivization of the linear map that subtracts x_1 from every other coordinate a maximal number of times and then permutes the coordinates according to the cycles $(1\,2\,\ldots\,n)$ and $(1\,n)$ respectively.

We prefer considering the map on an n-cover and a 2-cover where we do not reorder the coordinates but keep track of the ordering in another variable ϵ in $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively. Notice that the ergodic results we prove on such covers also apply to the map on the base.

As in the case of Brun algorithm, notice that, for $k > 1$, we compute the tree graph for some more general elementary operations on ordered vectors.

Proposition 3.8. *The win-lose induction defined by the following tree is conjugated, by the projection map defined in Equation* (1)*, to the map subtracting the* σ_k*-th coordinate from the* σ_{k-1}*-th coordinate of a vector ordered as* $x_{\sigma_1} > \cdots > x_{\sigma_n}$*.*



Assume we want to subtract from every other larger coordinate σ_k for some $k > 1$.

We start by subtracting it from its next largest coordinate σ_{k-1}. If the order is unchanged, we continue the process. If the order changes, then coordinate σ_k becomes coordinate σ'_{k-1} in the new ordering. We keep track of this coordinate by adding an index to the vertices of our graph, which is then labeled by $\sigma \in \mathfrak{S}_n$ and $k \in \{1, \ldots, n\}$. When $k = 1$, the coordinate cannot be subtracted from others anymore and we are done.

We can then apply the permutation to the coordinates. The coordinates which we subtract is given by $(1\,2\,\ldots\,n)^\epsilon \cdot 1$ for the Jacobi-Perron algorithm and $(1\,n)^\epsilon \cdot 1$ for the skew-product algorithm. The corresponding tree between domain vertices are represented in Figure 10.

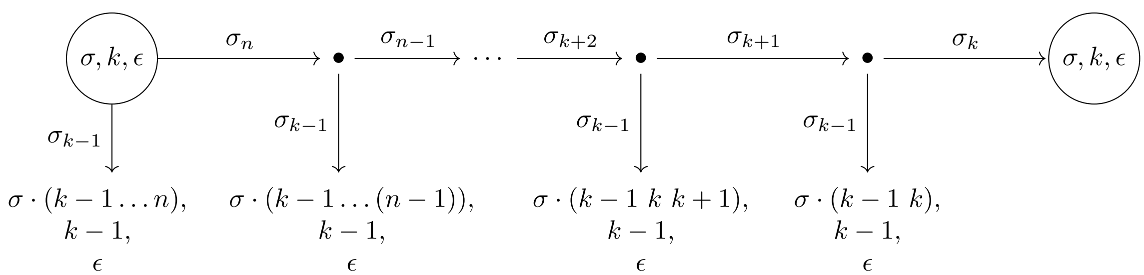


Figure 10: Detail of the graph for $k > 1$.

$$\boxed{\sigma, 1, \epsilon} = \begin{pmatrix} \sigma \\ \sigma^{-1} \cdot (1\,2\,\ldots\,n)^{\epsilon+1} \cdot 1 \\ \epsilon + 1 \end{pmatrix}$$

(a) Identification for $k = 1$ in Jacobi-Perron.

$$\boxed{\sigma, 1, \epsilon} = \begin{pmatrix} \sigma \\ \sigma^{-1} \cdot (1\,n)^{\epsilon+1} \cdot 1 \\ \epsilon + 1 \end{pmatrix}$$

(b) Identification for $k = 1$ in skew-product.

Notice that in this tree, σ_1 and σ_k are preserved. And after the identifications 11a (resp. 11b) where we denote by σ', k', ϵ' the new parameters for the leaf vertex after identification, $\sigma'_1 = \sigma_k$ and $\sigma'_{k'} = (1\,\ldots\,n)^{\epsilon'} \cdot 1$ (resp. $(1\,n)^{\epsilon'} \cdot 1$).

We then define the graph G on the following set of domain vertices connected by the previous tree, respectively

$$V_\circ = \left\{(\sigma, k, \epsilon) \in \mathfrak{S}_n \times \{2, \ldots, n\} \times \mathbb{Z}/n\mathbb{Z} \mid \sigma_1 = (1\ \ldots\ n)^{\epsilon-1} \cdot 1 \text{ and } \sigma_k = (1\ \ldots\ n)^{\epsilon} \cdot 1\right\}$$

and

$$V_\circ = \left\{(\sigma, k, \epsilon) \in \mathfrak{S}_n \times \{1, \ldots, n\} \times \mathbb{Z}/2\mathbb{Z} \mid \sigma_1 = (1\, n)^{\epsilon-1} \cdot 1 \text{ and } \sigma_k = (1\, n)^{\epsilon} \cdot 1\right\}.$$

In both cases, we define the projection

$$\pi : \begin{cases} \mathbb{R}_+^n \times V_\circ \longrightarrow \left\{x \in \mathbb{R}_+^n \mid x_n \geq x_i \text{ for all } 1 \leq i \leq n\right\} \\ (x, (\sigma, k, \epsilon)) \longmapsto (1\ \ldots\ n)^{\epsilon} \cdot \Pi_\sigma(x) \quad (\text{resp. } (1\, n)^{\epsilon} \cdot \Pi_\sigma(x)) \end{cases}.$$

Where Π_σ is the linear projection map defined in Equation (1) for each ordering $\sigma \in \mathfrak{S}_n$. Let p be the induced projection on the projectivized space by π. The image of p is the definition domain of F.

The algorithms are both iterations of these win-lose algorithms until ϵ changes. To see them as a first return map, let us consider $G^1 = (V^1, E^1)$ the 1-path graph associated to $V_\circ$. Recall that there is a projection $\pi_1 : G^1 \to G$.

Let W be the subset of vertices in V^1 associated to a path $\gamma_1\gamma_f$ where γ_1 goes from (σ, k, ϵ) to (τ, t, δ) in V^1 and $\epsilon \neq \delta$. Notice that necessarily, $\delta = \epsilon + 1$, $k = 2$, $\tau_t = (1\ \ldots\ n)^{\delta} \cdot 1$ and $\tau_1 = \sigma_2 = (1\ \ldots\ n)^{\epsilon} \cdot 1$.

Proposition 3.9. *Consider the win-lose induction defined by the graph G^1. Let $\Theta_\circ$ be its first return map to vertices in W and $T_\circ$ the induced projectivized map. Then we have $(p \circ \pi_1) \circ T_\circ = F \circ (p \circ \pi_1)$.*

Proposition 3.10. *The win-lose graphs in both cases are non-degenerating.*

Proof. By Lemma 1.5, one only needs to prove it for graphs G. As for Brun algorithm, starting at the vertex associated with permutation $\sigma \in \mathfrak{S}_n$ and integer k, win-lose induction goes to another vertex associated to a permutation $\sigma(k-1 \ldots l)$ where σ_l is the last winning label. In other terms, we insert label σ_{k-1} at the last winning label position. In $G_{\mathcal{L}}$, the following quantity does not decrease

$$M(\sigma, \mathcal{L}) := \max\{0 \leq i \leq n \mid \sigma([n-i+1, n]) \subset \mathcal{L}\}.$$

Again, consider the part of the degenerate subgraph $G_{\mathcal{L}}$ in the tree described on Figure 10 for a permutation σ and $k \geq 2$.

If $\sigma_{k-1} \notin \mathcal{L}$, there are no vertices with more than one edge labeled in $\mathcal{L}$.

If $\sigma_{k-1} \in \mathcal{L}$, let $l = M(\sigma, \mathcal{L})$. Notice that if $k = 2$ then $l \leq n-2$ and in particular $n - l \geq k$.

- If $n-l \geq k$, for vertices in the path composed of edges labeled by $\sigma_n, \sigma_{n-1}, \ldots, \sigma_{n-l+1}$ in $\mathcal{L}$, consider the subpath connecting it to the end vertex and add the edge labeled by σ_{k-1}. It points to a vertex labeled by $\sigma\cdot(1 \ldots (n-l))$. As $M(\sigma\cdot(1 \ldots (n-l)), \mathcal{L}) > l$ this path is labeled in $\mathcal{L}$ and leaves the strongly connected component of the initial vertex.

  On the other hand, black vertices in the tree that are not contained in the previous path, have a trivial strongly connected component.

- If $n-l<k$, follow the path labeled by σ_{k-1} which decreases k. As long as $n-l<k$ and $\sigma_{k-1}\in\mathcal{L}$ follow the edge σ_{k-1}. If the new permutation satisfies $\sigma_{k-1}\notin\mathcal{L}$, since $\sigma_n,\dots,\sigma_k\in\mathcal{L}$, the subgraph of the tree in Figure 10 for σ is a simple loop in $G_{\mathcal{L}}$, which is a strongly connected component distinct from the component of the initial vertex. If the process stops because $n-l\geq k\geq 2$ we follow the same path as in the previous case which decreases $M(\sigma,\mathcal{L})$. Again this is a path labeled in $\mathcal{L}$ which leaves the initial strongly connected component.

We finish by proving strong connectivity of G for Jacobi-Perron case, the proof is similar for skew-products. First, notice that for $\sigma,\sigma'\in\mathfrak{S}_n$ and $\epsilon\in\mathbb{Z}/n\mathbb{Z}$, there exists a path in the tree from (σ,n,ϵ) to $(\sigma',1,\epsilon)$ if and only if $\sigma'_1=\sigma_n$. Or more generally, there exists a path in the tree from (σ,k,ϵ) to $(\sigma',1,\epsilon)$ if and only if $\sigma'_1=\sigma_k$ and the relative order of $\sigma_k,\sigma_{k+1},\dots,\sigma_n$ is preserved in σ'. This is simply given by an insertion sorting algorithm. In particular, there is a path from any $(\sigma,n,\epsilon)\in V_\circ$ to all $(\sigma',n,\epsilon')\in V_\circ$.

Let $(\sigma,k,\epsilon)\in V_\circ$ with $k<n$ and l such that $\sigma_l=(1\,\dots\,n)^{\epsilon+1}\cdot 1$ the order of the letter which will be subtracted next. Recall that $\sigma_k=(1\dots n)^{\epsilon}\cdot 1$, then $l\neq k$.

- If $l>k$, let $\sigma'=\sigma\cdot(1\,\dots\,k)^{-1}$ and $l'=l$.
- If $l<k$, let $\sigma'=\sigma\cdot(l\,\dots\,n)\cdot(1\,\dots\,k-1)^{-1}$ and $l'=n$.

In both cases, $\sigma'_1=\sigma_k$ and $\sigma_k,\dots,\sigma_n$ preserve their relative order in σ'. So there is a path from (σ,k,ϵ) to $(\sigma',l',\epsilon+1)$ where $l'>k$. Hence, there exists $\sigma'\in\mathfrak{S}_n$ and $\epsilon'\in\mathbb{Z}/n\mathbb{Z}$ such that there is a path in G between (σ,k,ϵ) and (σ',n,ϵ').

Let (τ,t,δ) in $V_\circ$ there exists a path from (σ,k,ϵ) to some vertex (σ',n,ϵ'). Moreover, let us define $\tau'=\tau\cdot(k\,\dots\,n)^{-1}\cdot(k+1\,\dots\,n)^{-1}\dots(n-1\,n)^{-1}$. Then $\tau'_n=\tau_k$ thus there is a path from (τ',n,δ) to (τ,k,δ). And we have seen that there is a path from (σ',n,ϵ') to (τ',n,δ). □

3.3 Bounded coefficients Fractal sets

There is a rich literature on the set of real numbers with bounded continued fractions and its generalization to multidimensional continued fractions (see for instance [JP01] or [BL23] on Ostrowski algorithm). We explain in the following how such fractal sets can be described in the framework of this article and satisfy non-degenerating property. This implies in particular bounds on their Hausdorff dimension which can be computed numerically. We state a general consequence on these fractal sets first for Brun multiplicative algorithms then for Jacobi-Perron and Skew-product algorithms.

Theorem 3.11. *In multiplicative Brun, Jacobi-Perron and Skew-product algorithms the following holds. For all $N>0$ the Hausdorff dimension of the set of parameters which integer coding is bounded by N is strictly smaller than the dimension of the ambient space.*

As noticed in Remark 1.4, the fractal set of numbers which expansion is bounded by N can be described using the subgraph F^N of G^N, where we do not loop on vertices corresponding to a permutation more than $N-1$ times.

Namely, for the three cases we consider, if a vertex in G^N corresponds to a path $\gamma_1\dots\gamma_1$ from a domain vertex associated to a permutation σ, we remove the last edge of the path looping on this vertex. This is represented on Figure 12 and Figure 13 where the edges are labeled by the permutation corresponding to there image by π_N.

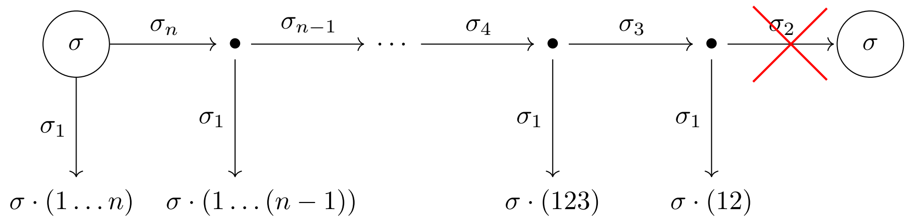


Figure 12: Removed edge in the tree corresponding to a path of shape $\gamma_1 \dots \gamma_1$ in F^N for Brun.

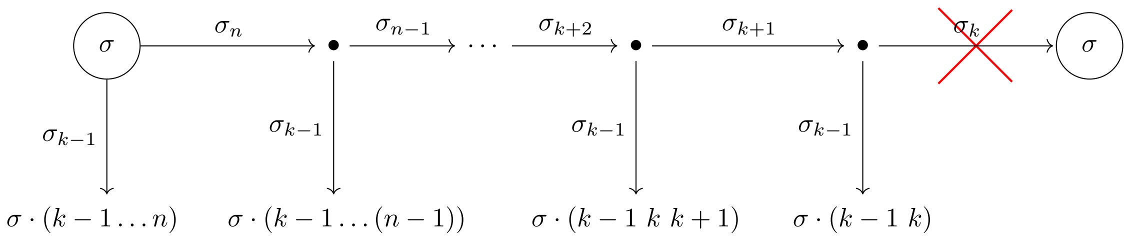


Figure 13: Removed edge in the tree corresponding to a path of shape $\gamma_1 \dots \gamma_1$ in F^N for Jacobi-Perron and Skew-products.

Proposition 3.12. *The subgraphs F^N are admissible.*

By Theorem B, this implies in particular Theorem 3.11.

Proof. As in Proposition 3.6 and Proposition 3.10, if we take $F_{\mathcal{L}}$, the induction on fibers of vertices corresponding to a permutation σ does not decrease the quantity $M(\sigma, \mathcal{L}) := \max\{0 \le i \le n \mid \sigma([n-i+1, n]) \subset \mathcal{L}\}$. Let us prove the proposition for Jacobi-Perron, the other cases are closely similar.

If $\sigma_{k-1} \notin \mathcal{L}$ then for all edges except for the last right bullet vertex, we clearly have either Property 2 or 3 depending on whether the horizontal edge is in $\mathcal{L}$ or not. For that last vertex, if $\sigma_k \in \mathcal{L}$ then along the branch path labeled by σ_{k+1} and σ_{k-1} the letter σ_k wins against σ_{k-1} which is not in $\mathcal{L}$ thus the edge is not in $F_{\mathcal{L}}$. If $\sigma_k \notin \mathcal{L}$, the action of the second edge of the path does not change coordinates in $\mathcal{L}$, hence the branch path satisfies Property 3.

If $\sigma_{k-1} \in \mathcal{L}$, we have constructed in the proof of the non-degenerating property in Proposition 3.10 a path labeled in $\mathcal{L}$ which leaves the strongly connected component $\mathscr{C}$ starting at any vertex. Since the removed edge in F does not appear in this path, this prove that Property 1 holds.

Strong connectivity of G implies it on G^N and F^N since we can find a path that connects the start and end vertices of a removed edge: follow the vertical edge and go back to the σ, then loop on σ until the path comes to a vertex of the from $\gamma_1 \dots \gamma_1$. $\square$

3.4 Selmer algorithms

Introduced by Selmer in 1961 [Sel61], the Selmer algorithm in dimension 3 is defined by the map

$$\Phi : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \to (x'_1, x'_2, x'_3) \in \mathbb{R}^3_+,$$

where, for any permutation $\{i, j, k\} = \{1, 2, 3\}$ and assuming $x_i > x_j > x_k$, we have:

$$x'_i = x_i - x_k, \quad x'_j = x_j, \quad x'_k = x_k.$$

Figure 14 illustrates the action of the Selmer algorithm on its definition domains. Unlike the Brun algorithm, the image sets here are not unions of domain sets. This difference arises from the fact that the subcone D, defined by the condition $x_i < x_j + x_k$ for all $\{i, j, k\} = \{1, 2, 3\}$, is an invariant subset of this map.

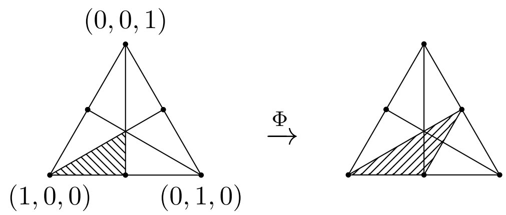


Figure 14: Action on definition domains

The Selmer algorithm does not alter the order of coordinates in the region $\mathbb{R}^n_+ \setminus D$, implying that every point in this complementary set eventually maps to D after a finite number of iterations. As a result, we can restrict our analysis to the map $\Phi_{|D}$.

Restricted to D, the algorithm has a straightforward description. Assume that $\max(x_2, x_3) \leq x_1 \leq x_2 + x_3$, then:

$$\Phi(x_1, x_2, x_3) = \begin{cases} (x_1 - x_3, x_2, x_3) & \text{if } x_2 > x_3 \quad \text{and } x_2 > \max(x_1 - x_3, x_3) \\ (x_1 - x_2, x_2, x_3) & \text{if } x_3 > x_2 \quad \text{and } x_3 > \max(x_1 - x_2, x_2) \end{cases}.$$

Other cases are defined similarly, by conjugating the map with a permutation of coordinates to reduce to the above cases.

In Figure 15, we represent the projective action of the restriction of the Selmer algorithm on D. Notice that the cone D is the convex hull of positive rays generated by vectors $v_1 = (0, 1, 1)$, $v_2 = (1, 0, 1)$ and $v_3 = (0, 1, 1)$. There are only 3 different image sets I_a, I_b, I_c described in the right part of the figure and they partition D. Thus we choose to describe the combinatoric graph for these sets, reducing the number of vertices to 3.

In Figure 15, we depict the projective action of the Selmer algorithm restricted to D. Note that the cone D is the convex hull of the positive rays generated by the vectors $v_1 = (0, 1, 1)$, $v_2 = (1, 0, 1)$, and $v_3 = (1, 1, 0)$. There are only three distinct image sets, I_a, I_b, I_c, as shown on the right side of the figure, and they partition D. To simplify, we describe the combinatoric graph for these sets, reducing the number of vertices to three.

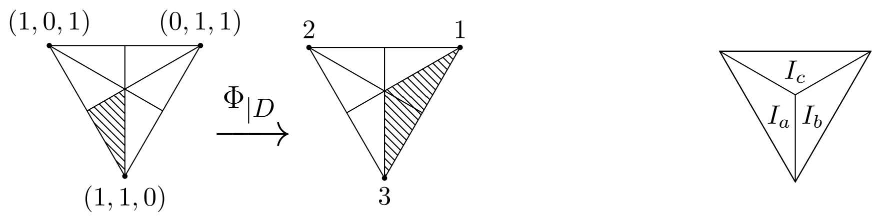


Figure 15: Projective action on D and image sets

The image sets are the projections of the positive cone $\mathbb{R}^n_+$ under the following matrices:

$$\Pi_a = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \Pi_b = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \Pi_c = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The corresponding win-lose induction is given by the graph on Figure 16.

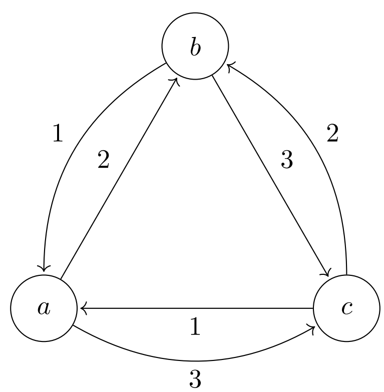


Figure 16: Selmer algorithm as a win-lose induction

Proposition 3.13. *For Θ the win-lose induction defined on Figure 16, $\pi \circ \Theta = \Phi_{|D} \circ \pi$.*

Let us consider the generalization of this algorithm for $n \geq 3$,

$$\Phi : (x_1, \dots, x_n) \in \mathbb{R}^n_+ \to (x'_1, \dots, x'_n),$$

where if $\sigma \in \mathfrak{S}_n$ is such that $x_{\sigma_1} > \cdots > x_{\sigma_n}$, we define

$$\begin{aligned} x'_{\sigma_1} &= x_{\sigma_1} - x_{\sigma_n} \\ x'_{\sigma_i} &= x_{\sigma_i} \text{ for all } i \geq 2. \end{aligned}$$

Similarly to the case of dimension 3, there is a stable subsimplex D defined such that for all $\sigma \in \mathfrak{S}_n$, if $x_{\sigma_1} > \cdots > x_{\sigma_n}$ then $x_{\sigma_1} < x_{\sigma_{n-1}} + x_{\sigma_n}$. On the complement of D, the map preserves the fact that x_{σ_n} is the smallest coordinate. Thus, every orbit enters D in finite time, reducing the study to the map $\Phi_{|D}$.

As for Brun algorithms, the domains of definition inside D are labeled by $\sigma \in \mathfrak{S}_n$ and will be denoted by D_σ. The image set $\Phi(D_\sigma) \subset D$ is the set of points satisfying $x'_{\sigma_2} > \cdots > x'_{\sigma_n}$ and $x'_{\sigma_1} < x'_{\sigma_{n-1}}$. The coordinate x'_{σ_1} can either occupy position $n-1$ or n, implying:

$$\Phi(D_\sigma) = \overline{D_{\sigma\cdot(1\ldots(n-1))}} \cup \overline{D_{\sigma\cdot(1\ldots n)}}.$$

Thus, the corresponding combinatoric graph has edges pointing from σ to $\sigma \cdot (1 \ldots (n-1))$ and $\sigma \cdot (1 \ldots n)$.

For $\sigma \in \mathfrak{S}_n$, let us denote by

$$v^i_\sigma = 11 \ldots 1\underset{\sigma_i}{0}1 \ldots 1,$$

the vector which has value 1 at all coordinates except σ_i at which it is 0. Such a vector is an extremal ray of the cone D and is fixed by Φ.

For $1 \le k \le n-2$, let us define the vector w^k_σ defined for all $1 \le i \le n$ by

$$w^k_\sigma(i) = \begin{cases} 2 & \text{if } \ i = \sigma_1, \ldots, \sigma_k \\ 1 & \text{otherwise.} \end{cases}$$

Let c be the vector with all coordinates equal to 1. The set D_σ is the convex hull of rays generated by $w^1_\sigma, \ldots, w^{n-2}_\sigma, v^n_\sigma$ and c, as there are d extremal points of the convex set, and D_σ is defined by d inequalities.

The projection map from $\mathbb{R}^{\mathcal{A}}_+$ to D_σ, denoted Π_σ, is defined by its columns $\{\Pi^i_\sigma\}_{1,\ldots,d}$. These vectors as well as their image under Φ are defined as follows:

$$\begin{array}{llll} \Pi^{\sigma_1}_\sigma & := b & \mapsto & v^1_\sigma = v^n_{\sigma\cdot(1\ldots n)} \\ \Pi^{\sigma_2}_\sigma & := w^1_\sigma & \mapsto & b \\ \Pi^{\sigma_{i+1}}_\sigma & := w^i_\sigma & \mapsto & w^{i-1}_{\sigma\cdot(1\ldots n)} = w^{i-1}_{\sigma\cdot(1\ldots(n-1))} \qquad \text{for } 2 \le i \le n-2 \\ \Pi^{\sigma_n}_\sigma & := v^n_\sigma & \mapsto & v^n_\sigma = v^n_{\sigma\cdot(1\ldots(n-1))} \end{array}$$

Notice that the image of $b + v^n_\sigma$ is given by $w^{n-2}_{\sigma\cdot(1\ldots(n-1))} = w^{n-2}_{\sigma\cdot(1\ldots n)}$. The images of the above vector satisfy the following equalities with $\tau = \sigma\cdot(1\ldots n)$ and $\tau' = \sigma\cdot(1\ldots(n-1))$.

$$\begin{array}{llllll} \Pi^{\sigma_1}_\sigma & \mapsto & \Pi^{\tau_n}_\tau & = \Pi^{\sigma_1}_\tau & & \\ \Pi^{\sigma_2}_\sigma & \mapsto & \Pi^{\tau_1}_\tau & = \Pi^{\sigma_2}_\tau & = & \Pi^{\sigma_2}_{\tau'} \\ \Pi^{\sigma_{i+1}}_\sigma & \mapsto & \Pi^{\tau_i}_\tau & = \Pi^{\sigma_{i+1}}_\tau & = & \Pi^{\sigma_{i+1}}_{\tau'} \quad \text{for } 2 \le i \le n-2, \\ \Pi^{\sigma_n}_\sigma & \mapsto & \Pi^{\tau_n}_{\tau'} & & = & \Pi^{\sigma_n}_{\tau'} \\ \Pi^{\sigma_1}_\sigma + \Pi^{\sigma_n}_\sigma & \mapsto & \Pi^{\tau_{n-1}}_\tau & = \Pi^{\sigma_n}_\tau & = & \Pi^{\sigma_1}_{\tau'} \end{array}$$

Consider the projection via Π_σ of the subcones of $\mathbb{R}^n_+$ where the coordinate corresponding to σ_1 is respectively smaller and larger than the one corresponding to σ_n. Hence, the Selmer map sends these subcones to D_τ and $D_{\tau'}$ respectively.

The graph associated to Selmer algorithm is then defined in Figure 17.

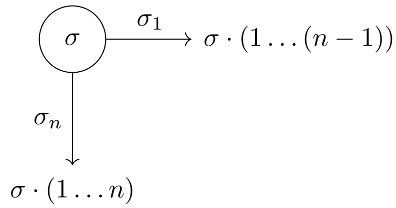


Figure 17: Win-lose tree between domain sets for Selmer algorithm

Proposition 3.14. *For Θ the win-lose induction defined on Figure 17, $\pi \circ \Theta = \Phi_{|D} \circ \pi$.*

Proposition 3.15. *The graph associated with the Selmer algorithm restricted to D in all dimensions is non-degenerating.*

Proof. Let us denote by G the graph in Figure 17. It is strongly connected since the permutation group is generated by the two cycles $(1 \dots n)$ and $(1 \dots (n-1))$.

Let $\mathcal{L}$ be a non-trivial subset of $\mathcal{A}$. The property that $\sigma_n \notin \mathcal{L}$ is preserved when following a path in the subgraph $G_{\mathcal{L}}$. Thus, in a strongly connected component, σ_n is either always or never in $\mathcal{L}$.

In a strongly connected component such that $\sigma_n \notin \mathcal{L}$, all vertices have one of its two outgoing edges not labeled in $\mathcal{L}$.

If $\sigma_n \in \mathcal{L}$, it remains so in the next step unless $\sigma_1 \notin \mathcal{L}$. But at each step the numbers $\sigma_1, \dots, \sigma_{n-1}$ are shifted to the left in the permutation. Hence, in less than n steps, the permutation will have σ_1 in $\mathcal{A} \setminus \mathcal{L}$. This produces a path labeled in $\mathcal{L}$ that leaves the strongly connected component. □

In particular, Theorem A provides an alternative proof of the following result.

Theorem ([Mö54], [Sch00])**.** *Selmer algorithms in all dimensions admit an invariant ergodic measure equivalent to Lebesgue measure.*

Again this theorem was proved by these authors for sorted algorithms. It is implied in our case by the following stronger result on unsorted algorithms.

Theorem 3.16. *Selmer algorithms in all dimensions admit a unique invariant ergodic measure equivalent to Lebesgue measure which induces the unique measure of maximal entropy of its canonical suspension flow.*

3.5 Rauzy Gaskets and Arnoux-Rauzy-Poincaré

Following [AS13], we define the Rauzy gasket in arbitrary dimension $n \geq 2$. Let $K = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n \mid x_j \leq \sum_{i \neq j} x_i,\ \forall j\}$ and the map defining Arnoux-Rauzy algorithm

$$\Phi : (x_1, \dots, x_n) \in \mathbb{R}_+^n \setminus K \to (x'_1, \dots, x'_n)$$

where if $\sigma \in \mathfrak{S}_n$ is such that $x_{\sigma_1} > \dots > x_{\sigma_n}$, we define

$$x'_{\sigma_1} = x_{\sigma_1} - \sum_{i=2}^{n} x_{\sigma_i}$$
$$x'_{\sigma_i} = x_{\sigma_i} \text{ for all } i \geq 2.$$

The Rauzy gasket $\mathcal{G}$ is the projective image in the simplex $\Delta = \mathbb{R}_+^n/\sim$, as defined in the first section, of the limit set

$$\mathcal{F} := \bigcap_{n\geq 0} \Phi^{-n}(\mathbb{R}_+^n \setminus K).$$

Let D_σ^K be the subsets of $\mathbb{R}_+^n \setminus K$ with coordinates ordered according to permutation σ. The projection matrices Π_σ^{-1} are defined by

$$z_{\sigma_n} = x_{\sigma_n},\ z_{\sigma_{n-1}} = x_{\sigma_{n-1}} - x_{\sigma_n},\ \dots\ ,\ z_{\sigma_2} = x_{\sigma_2} - x_{\sigma_3} \text{ and } z_{\sigma_1} = x_{\sigma_1} - x_{\sigma_2} - \cdots - x_{\sigma_n}.$$

Notice that the image sets are split by $\Phi(D_\sigma^K) = \bigcup_{\sigma\mathfrak{S}_n} \overline{D_\sigma}$ as for n-dimensional Brun algorithm. The splitting will then be similar, with an extra part of the tree to cut the part intersecting K in D_σ.

We start with the same tree given in Brun algorithm and replace the ending states by intermediate states $\widetilde{\sigma}$. At state $\widetilde{\sigma}$, in order to get D_σ^K, it remains to cut out of D_σ

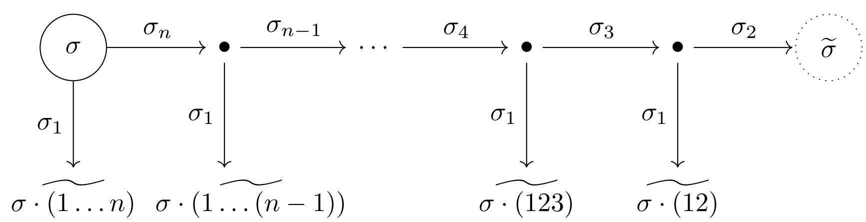


Figure 18: Win-lose tree for Rauzy gasket connecting σ to intermediate $\widetilde{\sigma}$ vertices

the subset $D_\sigma \cap K$ *i.e.* the set of points $(x_1, \dots, x_n)$ such that $x_{\sigma_1} > \cdots > x_{\sigma_n}$ and $x_{\sigma_1} < x_{\sigma_2} + \cdots + x_{\sigma_n}$. Applying the change of basis on D_σ defined in Equation (1) in Section 3.1 to send the subcone to $\mathbb{R}_+^n$, this reduces to equation

$$y_{\sigma_1} + \cdots + y_{\sigma_n} < y_{\sigma_2} + 2y_{\sigma_3} + \cdots + (n-1)y_{\sigma_n}$$
$$\iff y_{\sigma_1} < y_{\sigma_3} + 2y_{\sigma_4} + \cdots + (n-2)y_{\sigma_n}.$$

This subsimplex can be cut out of Δ by the following graph, where label of the horizontal vertices are σ_3 once, σ_4 twice, ... and σ_n $n-2$ times.

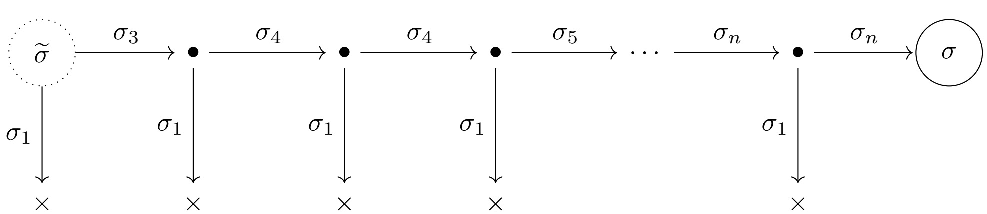


Figure 19: Part of the win-lose tree connecting $\widetilde{\sigma}$ to σ vertices for Rauzy gasket

Let F be the subgraph of the graph G defined in Figure 18 and Figure 19 to which we remove edges pointing to $\times$. We denote by $V_\circ$ the set of vertices labeled by $\sigma \in \mathfrak{S}_n$ (and not $\widetilde{\sigma}$). The projection matrices defined above corresponding to these vertices induce a bijection $\pi : \mathbb{R}_+^n \times V_\circ \to \mathbb{R}_+^n$ and a projectivized one $p : \Delta \times V_\circ \to \Delta$. The following proposition relates the induced fractal set on states $V_\circ$ with Rauzy gasket.

Proposition 3.17. *Consider the win-lose induction defined by the subgraph F in G and let $\Theta_\circ$ be its first return map to vertices in $V_\circ$. Let us denote by $\Delta_\circ^\infty$ the set $\Delta^\infty(F,G) \cap (\Delta \times V_\circ)$ Then $\pi(\Delta_\circ^\infty) = \mathcal{F}$, $p(\Delta_\circ^\infty) = \mathcal{G}$ and, on $\Delta_\circ^\infty$, $\Phi \circ \pi = \pi \circ \Theta_\circ$.*

For $\emptyset \subsetneq \mathcal{L} \subsetneq \mathcal{A}$, one would like to factor paths in the graph until the last edge is labeled in $\mathcal{L}$. To do so, we consider $G^1 = (V^1, E^1)$ the edge graph of G where vertices V^1 are given by E edges of G and two edges e and e' are connected by an edge labeled α if and only if there is an edge labeled α between their end vertex and start vertex respectively in G. It is a particular case of 1-path graph taking all vertices as state vertices.

One can define similarly a subgraph F^1 from which vertices with no outgoing edges are removed and $V_\circ^1$ vertices to which the corresponding edge points to $V_\circ$. Then the graph G^1, subgraph F^1 and set $V_\circ^1$ satisfy again the previous proposition.

We now consider $\widetilde{V}_{\mathcal{L}}^1$ the set of vertices corresponding to edges labeled in $\mathcal{L}$ and which ending vertex is either a bullet between σ and $\widetilde{\sigma}$ or a state vertex $\widetilde{\sigma}$.

Proposition 3.18. *There is no loop in the graph F restricted to $V^1 \setminus \widetilde{V}_{\mathcal{L}}^1$.*

Proof. Consider a $\mathcal{L}$-factor path in this restricted graph, assume it goes through a state vertex σ as in Figure 18 (which must happen after a finite number of steps). Then, it should leave the tree without going through an edge labeled by $\mathcal{L}$. In particular, $\sigma_1 \notin \mathcal{L}$ and the next permutation state σ' it goes through is such that $m(\sigma', \mathcal{L}) := \min\{0 \leq i \leq n \mid \sigma'(i) \in \mathcal{L}\}$ is strictly smaller that $m(\sigma, \mathcal{L})$. Hence the path cannot be a loop. □

In particular, a path γ in $G_{\mathcal{L}}$ between two vertices of $\widetilde{V}_{\mathcal{L}}^1$ can be uniquely decomposed as a finite concatenation $\gamma_1 \dots \gamma_n$ where each γ_i is a path with only its start and end vertices in $\widetilde{V}_{\mathcal{L}}^1$. By construction, it can meet at most one branching vertex with an outgoing edge labeled in $\mathcal{L}$ — the penultimate.

Proposition 3.19. *The family of subset of vertices $\{\widetilde{V}_{\mathcal{L}}^1\}_{\mathcal{L}}$ is an admissible factorization of F^1.*

Proof. Consider the subgraph $F_{\mathcal{L}}^1$. As in the proof of non-degenerating property for Brun algorithm, the accelerated induction goes from a permutation to the another and the quantity $M(\sigma, \mathcal{L}) := \max\{i \geq 0 \mid \sigma([n-i+1, n]) \subset \mathcal{L}\}$ does not decrease on them.

Let us consider γ a $\mathcal{L}$-factor path in a strongly connected component $\mathscr{C}$ of $F_{\mathcal{L}}$. Let σ the permutation corresponding to its starting vertex which is either $\widetilde{\sigma}$ of a bullet between σ and $\widetilde{\sigma}$.

If $\sigma_1 \in \mathcal{L}$, then the corresponding path is labeled in $\mathcal{L}$ and increases $M(\sigma, \mathcal{L})$. It thus leaves the strongly connected component $\mathscr{C}$ and γ satisfies Property 1 in Definition 1.3.

If $\sigma_1 \notin \mathcal{L}$ we show that the path satisfies Property 3.

If the path does not go through a state vertex σ', σ_1 is the only winning letter and is contained in labels of edges going out of the end vertex.

If the path γ goes through a state vertex $\widetilde{\sigma'}$, the label $\sigma'_1 \notin \mathcal{L}$ is the only one that may win against a label in $\mathcal{L}$. After it arrives at state σ', if the path visits a label in $\mathcal{L}$ before σ'_1 loses, σ'_1 is in the outgoing labels of the penultimate vertex. It then satisfies Property 3.

But if the path does not visit a label $\mathcal{L}$ then $M(\sigma', \mathcal{L}) = M(\sigma, \mathcal{L})$ must be 0. The quantity $m(\sigma', \mathcal{L}) = \min\{1 \leq i \leq n \mid \sigma'(i) \in \mathcal{L}\}$ is non-increasing in this component. Hence when σ'_1 loses, the path would leave the component, which brings a contradiction. □

As a consequence, Theorem B implies the following.

Theorem 3.20. *The Rauzy gasket in any dimension $n \geq 3$ has Hausdorff dimension strictly smaller than $n-1$ and its canonical suspension flow has a unique measure of maximal entropy.*

Arnoux–Rauzy–Poincaré The Arnoux-Rauzy-Poincaré algorithm is an algorithm in dimension $n = 3$ consisting in applying the Arnoux-Rauzy algorithm on $\mathbb{R}^3_+ \setminus K$ and the restriction of Poincaré algorithm on K (see [BL15]). It thus acts on K as described on Figure 20.

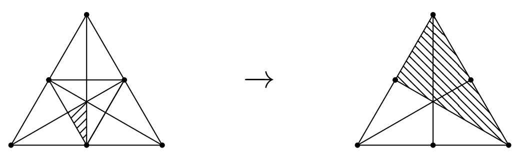


Figure 20: Action of Poincaré algorithm on a subdomain of K

We only need to make the edges pointing to the hole vertex $\times$ from $\widetilde{\sigma}$ point to $\sigma \cdot (123)$ as represented on Figure 21.

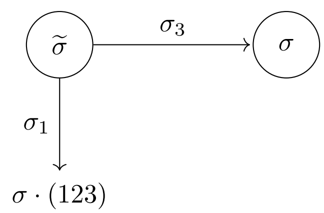


Figure 21: Connection for Arnoux-Rauzy-Poincaré in dimension 3

As for Brun algorithm in dimension 3, we only need to check the non-degenerating property for two letter subgraphs, say $G_{1,2}$. Here again the strongly connected components will be two loops around $1 > 2$ and $2 > 1$ formed by 3 edges. Which implies,

Proposition 3.21. *The graph associated to Arnoux-Rauzy-Poincaré algorithm in dimension 3 is non-degenerating.*

Theorem 3.22. *Arnoux-Rauzy-Poincaré algorithm admits a unique invariant ergodic measure equivalent to Lebesgue measure which induces the unique measure of maximal entropy of its canonical suspension flow.*

Observe that the generalization of this algorithm to higher dimensions will have more complicated combinatorics, since the images induced by the edges going out of the graph of Arnoux-Rauzy will produce new image sets. Perhaps another natural way to generalize this algorithm in the win-lose induction point of view would be to connect all these edges to some vertex $\sigma \cdot (1 \dots n)$. This again is a non-degenerating win-lose induction.

Link with Baragar constants For Rauzy gasket in dimension d, we show in this paragraph that the constant κ_0 in Theorem B is bounded from above by the constant computed in [Bar98] under notation $\alpha(d+1)$ and used in [GMR19] under notation $\beta(d+2)$.

Let us recall notations of [GMR19], we use the numbering in this article to refer to some results or formulas in the following paragraph. Let

$$\mathcal{H} := \left\{ (y_1, \dots, y_{n+1}) \in \mathbb{R}_+^{n+1} : y_1 \leq y_2 \leq \cdots \leq y_{n+1}, \sum_{j=1}^{n} y_j = y_{n+1} \right\},$$

$\mathcal{O} := \mathcal{H}/\mathbb{R}_+$ and Γ be the semigroup generated by $\gamma_1, \dots, \gamma_n$ maps on ordered $(n+1)-$ *tuples* defined for $1 \leq j \leq n$ by

$$\gamma_j(y_1, \dots, y_{n+1}) = \left(y_1, \dots, \hat{y_j}, \dots, y_{n+1}, \sum_{i \neq j} y_i \right).$$

As in Example 9, forgetting the $n+1$-th coordinate, these matrices correspond to inverse branches of Arnoux–Rauzy algorithm composed with a sorting permutation on coordinates. They consider the following set of product of these matrices

$$T_\Gamma = \{ \gamma_{k-1}^N \gamma_j \mid N \in \mathbb{Z}_{k \geq 0}, 1 \leq j \leq n-1 \}.$$

In formula (4.3) they introduce for all $s \in \mathbb{R}_+$ a transfer operator $\mathcal{L}_s : C^1(\mathcal{O}) \to C^1(\mathcal{O})$ defined for C^1 functions f on $\mathcal{O}$ and $x \in \mathcal{O}$ by

$$\mathcal{L}_s(f)(x) = \sum_{\gamma \in T_\Gamma} |Jac_w(\gamma)|^{-\frac{s}{n}} f(\gamma \cdot x)$$

where we have directly made the substitution induced by Lemma 44. Theorem 39 is a Ruelle–Perron–Frobenius theorem implying the existence of a positive real number λ_s which is the unique eigenvalue of this operator with the largest module. As remarked after Proposition 43, the number s such that $\lambda_s = 1$ corresponds to Baragar's constants.

We identify $\mathcal{O}$ with points $(x_1, \dots, x_n) \in \Delta = \mathbb{R}_+^n / \sim$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$. Consider the action of $\mathfrak{S}_n$ on Δ by permutation on the coordinates and its quotient map $\mathfrak{o} : \Delta \to \Delta/\mathfrak{S}_n \simeq \mathcal{O}$. Let Φ^* be the accelerated Arnoux-Rauzy algorithm composed until change of order. The images γx for $\gamma \in T_\Gamma$ are all points $y \in \mathcal{O}$ such that there exists $j \in \{1, \dots, n-1\}$ with $\Phi^*((j \dots n) \cdot y) =$. Hence they correspond to preimages for Φ^* on the whole *unordered* simplex Δ and these preimages are almost surely distinct. By an elementary computation, we have $|Jac_w(\gamma)|^{-\frac{1}{n}} = e^{-r(x)}$ for the roof function r defined in [Fou25]. We associate to f the pulled back map defined on Δ by $\widetilde{f} := f \circ \mathfrak{o}$. The transfer operator can then be expressed as

$$\mathcal{L}_s(f)(x) = \sum_{\Phi^*(y)=x} e^{-s \cdot r(x)} \widetilde{f}(y).$$

Let us consider the subset of $\Delta_\Gamma \subset \Delta \times V_\circ$ which corresponding path in the graph until the next image vertex is not a loop *i.e.*

$$\Delta_\Gamma := \bigcup_{\sigma \in V_\circ} (\Delta \setminus \Delta_{\gamma_\sigma}) \times \{\sigma\}$$

where $\gamma_\sigma = \sigma_n \cdot \sigma_{n-1} \cdot \dots \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4 \cdot \sigma_4 \cdot \sigma_5 \cdot \ldots \sigma_n$ is the path that loops on the vertex labeled by σ.

Let now $\Theta_\circ$ and $\Theta_\circ^*$ be the first return map of the win-lose induction on $\Delta \times V_\circ$ and Δ_Γ respectively. $\Theta_\circ$ is also conjugated to a first return map on vertices of the 1-path graph associated to the Arnoux-Rauzy graph.

These two maps are conjugated to Φ and Φ^* with respect to the map $p : \Delta \times V_\circ \to \Delta$ induced by the projectivization of the map π defined above in Section 3.1. For $\widehat{f} := \widetilde{f} \circ p$,

$$\sum_{p(\hat{y})=x} e^{-s \cdot r(x)} \widehat{f}(\hat{y}) = n \sum_{\Phi^*(y)=x} e^{-s \cdot r(x)} \widetilde{f}(y) = n\mathcal{L}_s(f)(x).$$

Thus

$$\sum_{p(\hat{y})=x} L_\phi(\widehat{f})(\hat{y}) = n\mathcal{L}_s(f)(x)$$

where $L_\phi : C(\Delta_\Gamma) \to C(\Delta_\Gamma)$ is the Ruelle operator studied in [Fou25] using thermodynamic formalism with potential function $\phi = -s \cdot r_*$. Even though it is not a first return on a subsimplex induced by a positive path, it is a uniformly expanding (Proposition 45) first return map on vertices and all the constructions of [Fou25] using thermodynamic formalism can be done.

Let us consider g an eigenfunction for L_ϕ on Δ_Γ and $\hat{x}$ a point in that space. Permutations act on these maps by $\sigma \cdot g(\hat{x}) := g(\sigma \cdot \hat{x})$. Notice that

$$L_\phi(\sigma \cdot g)(\hat{x}) = \sum_{\Theta_\circ(\hat{y})=\hat{x}} e^{-s \cdot r(\sigma \cdot \hat{x})} g(\sigma^{-1} \cdot \hat{y}) = \sum_{\Theta_\circ(\hat{y})=\sigma \cdot \hat{x}} e^{-s \cdot r(\sigma \cdot \hat{x})} g(\hat{y}) = L_\phi(g)(\sigma \cdot \hat{x}).$$

thus the action permutes with L_ϕ. And we can averaging over all permutations, to show eigenvalues of L_ϕ are also eigenvalues of $\mathcal{L}_s$.

As the C^0-norm is dominated by the C^1-norm, λ_s also bounds the module of eigenvalues of $\mathcal{L}_s$ extented to $C(\mathcal{O})$. The eigenvalue with largest module for L_ϕ is $e^{P_G(-s \cdot r_*)}$ and by the inclusion of eigenvalues proved above $e^{P_G(-s \cdot r_*)} \leq \lambda_s$. Moreover κ_0 is the unique value of s such that $P_G(-s \cdot r_*) = 0$ (and corresponds to the entropy of the canonical suspension flow). Hence $1 \leq \lambda_{\kappa_0}$ and, by Proposition 43, Baragar's constant is not smaller than κ_0. As a consequence of the computations in [Bar98] and Theorem B we have the following result.

Theorem 3.23. *If $\mathcal{G}^d$ denotes the Rauzy gasket in dimension d, we have the bounds*

$$\begin{aligned} \dim_H(\mathcal{G}^2) &< 1.825, \\ \dim_H(\mathcal{G}^3) &< 2.7, \\ \dim_H(\mathcal{G}^4) &< 3.612 \end{aligned}$$

and for d going to infinity

$$\dim_H(\mathcal{G}^d) < d - 1 + \frac{\log d}{\log 2 \cdot (d+1)} + o(d^{-1.58}).$$

This strengthens and generalizes the only previous known bound proved in [AHS16] to be $\dim_H(\mathcal{G}^2) < 2$. After the first preprints of this article, a sharper bound: 1.7415, has been proven by Pollicott-Sewell [PS23]

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